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Boundary RG flow associated with the AKNS soliton hierarchy

Vladimir A Fateev[1,2] **and Sergei L Lukyanov**[2,3]

[1] Laboratoire de Physique Théorique et Astroparticules, Université Montpellier II, Pl E Bataillon, 34095 Montpellier, France
[2] L D Landau Institute for Theoretical Physics, Chernogolovka 142432, Russia
[3] NHETC, Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

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Abstract
We introduce and study an integrable boundary flow possessing an infinite number of conserving charges which can be thought of as quantum counterparts of the Ablowitz, Kaup, Newell and Segur Hamiltonians. We propose an exact expression for overlap amplitudes of the boundary state with all primary states in terms of solutions of certain ordinary linear differential equations. The boundary flow is terminated at a nontrivial infrared fixed point. We identify a form of whole boundary state corresponding to this fixed point.

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Contents

1. Introduction

The so-called hairpin model of boundary interaction was introduced in [1]. This two-dimensional model of quantum field theory (QFT) involves a two-component Bose field $\mathbf{X}(\sigma,\tau) = (X(\sigma,\tau), Y(\sigma,\tau))$ which lives on the semi-infinite cylinder $\tau \equiv \tau + 2\pi R$, $\sigma \geqslant 0$. In the bulk, $\sigma > 0$, the field $\mathbf{X}$ is a free massless field, as described by the bulk action

$$\mathscr{A}_{\text{bulk}} = \frac{1}{\pi}\int_0^{2\pi R} \mathrm{d}\tau \int_0^\infty \mathrm{d}\sigma\, \partial\mathbf{X}\cdot\bar\partial\mathbf{X}, \tag{1}$$

with $\partial = \frac{1}{2}(\partial_\sigma - \mathrm{i}\partial_\tau)$ and $\bar\partial = \frac{1}{2}(\partial_\sigma + \mathrm{i}\partial_\tau)$. The boundary values of this field, $\mathbf{X}|_{\sigma=0} = (X_B, Y_B)$, are subjected to a nonlinear constraint

$$\exp\left(\frac{X_B}{\sqrt{n}}\right) - \cos\left(\frac{Y_B}{\sqrt{n+2}}\right) = 0, \tag{2}$$

where n is a positive parameter. A remarkable feature of the model is that it possesses an extended conformal symmetry with respect to certain W-algebra. The generating holomorphic, $W_s = W_s(\sigma + \mathrm{i}\tau)$, and antiholomorphic, $\bar W_s = \bar W_s(\sigma - \mathrm{i}\tau)$, currents of this algebra have spins $s = 2, 3, 4, \ldots$ and $s = -2, -3, -4, \ldots$ respectively. Among them there are spin-(± 2) currents which are components of the stress–energy tensor:

$$W_2 = -\partial X\partial X - \partial Y\partial Y + \frac{1}{\sqrt{n}}\partial^2 X, \qquad \bar W_2 = -\bar\partial X\bar\partial X - \bar\partial Y\bar\partial Y + \frac{1}{\sqrt{n}}\bar\partial^2 X. \tag{3}$$

The first nontrivial holomorphic current reads explicitly as follows,

$$W_3 = \frac{6n+4}{3}(\partial Y)^3 + 2n(\partial X)^2\partial Y + n\sqrt{n}\partial^2 X\partial Y - (n+2)\sqrt{n}\partial X\partial^2 Y + \frac{n+2}{6}\partial^3 Y, \tag{4}$$

while the higher currents $W_4, W_5, \ldots$ can be generated recursively from the singular parts of operator product expansions of the lower currents. The antiholomorphic currents $\bar W_s$ can be obtained from the corresponding holomorphic one by means of the formal substitution $\partial \to \bar\partial$.

As usual in QFT, an effect of the boundary can be described in terms of the boundary state which incorporates all information about boundary conditions [2–5]. In our case the boundary state $|B\rangle_{\mathrm{hair}}$ is a special vector in the space of states $\mathcal{H}$ of the two-component uncompactified scalar associated with the 'equal time section' $\sigma = \mathrm{const}$:

$$|B\rangle_{\mathrm{hair}} \in \mathcal{H} = \int_{\mathbf{P}} \mathcal{F}_{\mathbf{P}} \otimes \bar{\mathcal{F}}_{\mathbf{P}}, \tag{5}$$

where $\mathcal{F}_{\mathbf{P}}(\bar{\mathcal{F}}_{\mathbf{P}})$ is the Fock space of two-component right-moving (left-moving) boson with the zero-mode momentum $\mathbf{P} = (P, Q)$. The above-mentioned W-invariance of the boundary condition (2) implies that the corresponding boundary state obeys an infinite set of equations [3, 4]:

$$[W_{s+1}(\tau) - \bar{W}_{s+1}(\tau)]_{\sigma=0}|B\rangle_{\mathrm{hair}} = 0. \tag{6}$$

Once the conformal symmetry is preserved, the hairpin boundary condition can be treated as a renormalization group (RG) fixed point in the space of boundary interactions of two-component free Bose field. Broadly speaking, any relevant boundary perturbation breaks down the scale invariance of the original model and introduces some RG invariant 'physical scale' E_* in the theory. Unfortunately, there is no systematic machinery for studying an arbitrary perturbation. Therefore, it makes sense to consider a particular class of perturbations preserving some amount of the original W-symmetry. The boundary state for such models satisfies the conditions

$$(\mathbb{I}_s - \bar{\mathbb{I}}_s)|B\rangle_{\mathrm{pert}} = 0 \tag{7}$$

for some operator-valued functionals $\mathbb{I}_s$ ($\bar{\mathbb{I}}_s$) of the original holomorphic (antiholomorphic) W-currents of the form

$$\mathbb{I}_s = \int_0^{2\pi R} \frac{\mathrm{d}\tau}{2\pi} P_{s+1}, \qquad \bar{\mathbb{I}}_s = \int_0^{2\pi R} \frac{\mathrm{d}\tau}{2\pi} \bar{P}_{s+1}, \tag{8}$$

where the densities $P_{s+1} = P_{s+1}[W_2, W_3, \ldots]$, $\bar{P}_{s+1} = P_{s+1}[\bar{W}_2, \bar{W}_3, \ldots]$ are appropriately regularized polynomials in W-currents and their derivatives. Here the subscript $s + 1$ labels the spin of the local field P_{s+1}. Roughly speaking, the meaning of (7) is that the boundary neither emits nor absorbs any amount of the combined charges $\bar{\mathbb{I}}_s - \mathbb{I}_s$. For this reason we shall call $\mathbb{I}_s$ a local integral of motion (IM) of spin s. Note that in the case

$$P_2 = W_2, \tag{9}$$

where W_2 is the holomorphic component of the stress–energy tensor, equation (7) manifests the invariance with respect to translations along the τ-direction.

Let us assume that an infinite sequence of polynomials $\{P_{s+1}\}$, such that the associated IM are mutually commutative,

$$[\mathbb{I}_s, \mathbb{I}_{s'}] = 0, \tag{10}$$

is given. It is natural to expect that a theory possessing such an infinite commuting set is integrable [5].

At the best of our knowledge the complete algebraic classification of infinite commuting sets of local IM for the hairpin W-algebra has not been obtained yet. Nevertheless, at least three nontrivial examples are known [6–8]. In [1] it was studied the RG boundary flow associated with the so-called paperclip series of local IM. The series contains local IM with the odd spins $s = 1, 3, 5, \ldots$. The second known Abelian subalgebra [7] is deeply related to the Ablowitz, Kaup, Newell and Segur (AKNS) soliton hierarchy [9, 10]. In fact the corresponding $\mathbb{I}_s$ are the quantum counterparts of the AKNS Hamiltonians. For this reason we shall refer to this infinite sequence of commuting integrals as AKNS series. Among characteristic properties of this series is that it contains the local IM with $s = 1, 2, 3, \ldots$ and the first two local densities are given by equation (9) and

$$P_3 = \mathrm{i}W_3. \tag{11}$$

One more series of the local IM containing the odd spins only is known. Despite admissible spins of $\mathbb{I}_s$ for this series and for the paperclip series are the same, they are not equivalent. An explicit form of $\mathbb{I}_3$ from the third series can be found in [6].

In this paper, we study an integrable model associated with the AKNS series of local IM. The theory can be defined by adding some special boundary term of the form

$$\mathscr{A}_{\mathrm{pert}} = \int_0^{2\pi R} \frac{\mathrm{d}\tau}{2\pi} U(\mathbf{X}_B) \tag{12}$$

to the bulk action (1). The potential $U(\mathbf{X}_B)$ turns out to be unbounded and purely imaginary. In spite of these somewhat pathological properties, the corresponding QFT appears to be well defined and possesses many remarkable features. We shall refer to this theory as the integrable perturbed hairpin (IPH) model[4].

As was pointed out in [11] a boundary state associated with integrable boundary flows with conformal bulk can be studied in a framework of quantum inverse scattering method [12–14]. To recall the basic idea by the example of IPH model let us assume that the corresponding local IM are Hermitian operators and the set $\{\mathbb{I}_s\}_{s=1}^{\infty}$ is 'resolving', i.e., that all eigenspaces of the local IM are one-dimensional and mutually orthogonal (which seems to be the case, see section 4.2). Then, as it follows from the structure of space $\mathcal{H}$ (5) and condition (7), the IPH boundary state can be written as

$$|B\rangle_{\mathrm{iph}} = \int_{\mathbf{P}} \mathrm{d}^2\mathbf{P} \sum_{\alpha} B_\alpha(\mathbf{P}) |\alpha, \mathbf{P}\rangle \otimes \overline{|\alpha, \mathbf{P}\rangle}, \tag{13}$$

where $\{|\alpha, \mathbf{P}\rangle\}$ is the orthonormalized basis of eigenvectors in each Fock space $\mathcal{F}_{\mathbf{P}}$ labelled by some index α. In view of equation (13), it is convenient to think of the boundary state in terms of the associated *boundary operator*. The natural isomorphism between $\mathcal{F}_{\mathbf{P}}$ and $\bar{\mathcal{F}}_{\mathbf{P}}$ (the right movers are replaced by the left movers) makes it possible to establish one-to-one correspondence between states in $\mathcal{F}_{\mathbf{P}} \otimes \bar{\mathcal{F}}_{\mathbf{P}}$ and operators in $\mathcal{F}_{\mathbf{P}}$. Thus the boundary state $|B\rangle_{\mathrm{iph}}$ can be reinterpreted as an operator

$$\mathbb{B} = \int_{\mathbf{P}} \mathrm{d}^2\mathbf{P} \sum_{\alpha} B_\alpha(\mathbf{P}) |\alpha, \mathbf{P}\rangle\langle\alpha, \mathbf{P}|, \tag{14}$$

which commutes with the all local IM:

$$[\mathbb{B}, \mathbb{I}_s] = 0. \tag{15}$$

Structure (13) emphasizes an importance of the problem of simultaneous diagonalization of local IM $\mathbb{I}_s$ as operators acting in $\mathcal{F}_{\mathbf{P}}$. In this connection, it is pertinent to remind that physical quantities such as the boundary state (operator) essentially depend on some RG invariant scale E_*. For the IPH model it is convenient to choose this dependence in the form $\mathbb{B} = \mathbb{B}(\lambda)$, where the dimensionless parameter λ is related to this scale by

$$\lambda = \left(\frac{E_* R}{n}\right)^{\frac{n+2}{n}}. \tag{16}$$

At the same time the associated local IM do not involve any particular energy scale and their eigenvectors, $\{|\alpha, \mathbf{P}\rangle\}$, do not depend on λ. Hence the operators $\mathbb{B}$ (14) corresponding to different values of the 'spectral' parameter λ commute between themselves:

$$[\mathbb{B}(\lambda), \mathbb{B}(\lambda')] = 0. \tag{17}$$

[4] Because of the unbound property of $U(\mathbf{X}_B)$ the term 'perturbed', here and below, is used in a loose sense.

The problem of simultaneous diagonalization of commuting operator families is a typical problem for the quantum inverse scattering method [12–14]. In this approach commuting families, such as $\mathbb{B}(\lambda)$ (17), are defined in terms of certain monodromy matrices associated with an auxiliary linear problem where λ plays a role of spectral parameter. In section 2, we briefly discuss the IPH model in the classical limit to clarify its relation to the AKNS soliton hierarchy. Later, in section 8.2, we shall present arguments that the boundary operator $\mathbb{B}(\lambda)$ in the quantum theory can be treated as a version of Baxter's Q-operator [15]. In particular, it satisfies the famous Baxter $T - Q$ equation:

$$\mathbb{B}(\lambda)\mathbb{T}(\lambda) = \mathbb{B}(\lambda\mathbf{q}) + \mathbb{B}(\lambda\mathbf{q}^{-1}) \qquad \left(\mathbf{q} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}}\right), \tag{18}$$

where the transfer matrix $\mathbb{T}(\lambda)$ is a trace of quantum 2×2 monodromy matrix for the auxiliary AKNS linear problem. The operator $\mathbb{T}(\lambda)$ can be thought of as a generating function for the AKNS series of local IM.

It should be emphasized that the paper does not contain a rigorous quantization procedure of the AKNS hierarchy. It is devoted to the study of the simplest boundary amplitude

$$Z = {}_{\mathrm{iph}}\langle B|\mathbf{P}\rangle, \tag{19}$$

where $|\mathbf{P}\rangle \in \mathcal{H}$ is the highest vector in the Fock modulus $\mathcal{F}_{\mathbf{P}} \otimes \bar{\mathcal{F}}_{\mathbf{P}}$ corresponding to an arbitrary $\mathbf{P}$. A wealth of data about Z can be obtained through perturbative analysis in the weak coupling domain, and by looking into various limiting cases of the model; sections 3–6 are devoted to these tasks. Using these data we propose in section 7 an exact expression for the vacuum overlap (19) in terms of solutions of certain ordinary differential equation. Only in section 8, examining properties of the vacuum amplitude Z, we reveal general integrable structures, such as Baxter $T - Q$ equation (18), inherent in the theory. We conclude the paper with a discussion of an infrared fixed point of the boundary flow.

2. The classical IPH model

Before going over to QFT, we will explore the classical limit of the model. For this purpose it is convenient to use the field

$$\mathbf{x} = (x, y) = \frac{\mathbf{X}}{\sqrt{n}}. \tag{20}$$

Indeed, if one rewrites (1) in terms of $\mathbf{x}$ the parameter n appears in front of the Gaussian action. This allows one to interpret $\frac{n}{2\pi}$ as the inverse Plank constant.

2.1. *Integrable perturbation of the classical hairpin model*

As $n \to \infty$, x and y become classical fields subjected by the boundary condition

$$[\mathrm{e}^{x} - \cos(y)]_{\sigma=0} = 0. \tag{21}$$

The classical equations of motion in the unperturbed hairpin model include the bulk equations, $\triangle x = \triangle y = 0$, as well as the boundary equation

$$\partial_\sigma \mathbf{x} \cdot \mathbf{t}|_{\sigma=0} = 0, \tag{22}$$

where $\mathbf{t} = (-\tan(y), 1)$ is a tangent vector to the curve (21). To take into account the zero mode, we shall consider the classical solutions $\mathbf{x}(\sigma, \tau)$ such that

$$\mathbf{x}(\sigma, \tau) \to \frac{2\mathrm{i}\boldsymbol{\xi}}{R}\sigma \quad \text{as} \quad \sigma \to +\infty, \tag{23}$$

where $\boldsymbol{\xi} = (\xi_x, \xi_y)$ is some constant vector.

In the large-n limit the quantum field W_3 (4) produces a classical holomorphic current, $W_3 \to n^{\frac{5}{2}} w_3$, with

$$w_3 = 2(\partial y)^3 + 2(\partial x)^2 \partial y + \partial^2 x \partial y - \partial^2 y \partial x, \tag{24}$$

and equation (6) for $s = 3$ implies that the difference $w_3 - \bar{w}_3$ vanishes at the boundary. One can indeed check that the last condition holds in virtue of the classical equations of motion and the boundary constraint (21).

Now let us analyse an effect of the boundary potential (12) in the classical theory. We still assume the boundary constraint (21), so the perturbation modifies the classical boundary equation of motion (22) only:

$$\partial_\sigma \mathbf{x} \cdot \mathbf{t}|_{\sigma=0} = f, \tag{25}$$

where $f = \frac{1}{\sqrt{n}} \nabla_{\mathbf{X}_B} U \cdot \mathbf{t}$. For an arbitrary function $U(\mathbf{X}_B)$ the holomorphic current w_3 does not generate a conserving charge because

$$[w_3 - \bar{w}_3]_{\sigma=0} = \frac{\mathrm{i}}{2}\left[-\frac{\mathrm{d}}{\mathrm{d}\tau}(f(x)\partial_\sigma x) + 3f^2 \cot(y)\partial_\tau x + 2\left(\frac{\mathrm{d}f}{\mathrm{d}x} + 2f\right)\partial_\tau x \partial_\sigma x\right]_{\sigma=0}. \tag{26}$$

However, if we adjust the boundary potential in such a way that (26) can be written in the form

$$[w_3 - \bar{w}_3]_{\sigma=0} = \mathrm{i}\frac{\mathrm{d}\theta_2}{\mathrm{d}\tau}, \tag{27}$$

with some boundary field $\theta_2 = \theta_2(\tau)$, then the charge

$$q_2 = \int_0^\infty \mathrm{d}\sigma (w_3(\tau,\sigma) + \bar{w}_3(\tau,\sigma)) - \theta_2(\tau), \tag{28}$$

will not depend on τ [5]:

$$\frac{\mathrm{d}q_2}{\mathrm{d}\tau} = 0. \tag{29}$$

This occurs for $f \sim \mathrm{e}^{-2x} = \frac{1}{\cos^2(y)}$, i.e., for the boundary potential such that

$$U \to -\frac{nC}{R}\tan(y) \qquad \text{as} \quad n \to \infty, \tag{30}$$

where C is an arbitrary dimensionless constant.

If one takes τ to be the Euclidean time, equation (29) signifies a presence of nontrivial conservation charge in the perturbed theory. As a matter of fact, the classical hairpin model with the boundary potential (30) possesses an infinite number of conserving charges q_s with $s = 1, 2, 3, \ldots$. They are generated by classical holomorphic currents w_{s+1} satisfying the condition

$$[w_{s+1} - \bar{w}_{s+1}]_{\sigma=0} = \mathrm{i}\frac{\mathrm{d}\theta_s}{\mathrm{d}\tau}. \tag{31}$$

For $s = 1$,

$$w_2 = -(\partial x)^2 - (\partial y)^2, \tag{32}$$

and the corresponding charge q_1 coincides with the energy. By means of a direct calculation it is not hard to find an explicit form of w_4:

$$w_4 = -(\partial x)^4 - 5(\partial y)^4 - 6(\partial x)^2(\partial y)^2 - 4\partial^2 x(\partial y)^2 + 4\partial^2 y \partial x \partial y + \partial^3 x \partial x + \partial^3 y \partial y. \tag{33}$$

It is useful to keep in mind simple ambiguities in a choice of w_{s+1} satisfying equation (31). First, these currents are defined up to total derivatives, $w_{s+1} \to w_{s+1} + \partial g$ with $\bar{\partial} g = 0$. Second, equation (31) does not fix an overall multiplicative normalization of w_{s+1}.

2.2. Local IM in the AKNS soliton hierarchy

Now we describe an effective way to generate all the densities w_{s+1} (31) up to the above-mentioned ambiguities. At this step we need to introduce the fields

$$\psi = \mathrm{i}(\partial y + \mathrm{i}\partial x)\exp(2\mathrm{i}y_R), \qquad \psi^* = \mathrm{i}(\partial y - \mathrm{i}\partial x)\exp(-2\mathrm{i}y_R), \tag{34}$$

where y_R in the exponential stands for the holomorphic part of the harmonic field y:

$$y(\tau,\sigma) = y_R(\sigma + \mathrm{i}\tau) + y_L(\sigma - \mathrm{i}\tau). \tag{35}$$

Hence ψ and ψ^* are non-locally expressed in terms of the fundamental field $\mathbf{x}$. They are clearly holomorphic ($\bar\partial\psi = \bar\partial\psi^* = 0$) and quasiperiodic fields:

$$\psi(\tau + 2\pi R) = \mathrm{e}^{-4\pi\mathrm{i}\xi_y}\psi(\tau), \qquad \psi^*(\tau + 2\pi R) = \mathrm{e}^{4\pi\mathrm{i}\xi_y}\psi^*(\tau), \tag{36}$$

where ξ_y is the second component of the vector $\boldsymbol{\xi}$ in equation (23).

The fields (34) are remarkable in many extents. First, the densities w_s in equations (24), (32), (33) are nicely expressed in terms of ψ and ψ^*:

$$w_2 = \psi\psi^*, \qquad w_3 = \frac{\mathrm{i}}{2}(\psi^*\partial\psi - \psi\partial\psi^*), \qquad w_4 = -\frac{1}{2}(\psi^*\partial^2\psi + \psi\partial^2\psi^*) - (\psi\psi^*)^2. \tag{37}$$

More importantly, ψ and ψ^* generate a closed Poisson subalgebra in the space of classical fields. To introduce the Hamiltonian picture here, we will interpret the world-sheet coordinate σ as the Euclidean time. Then the classical bulk action for the fundamental field $\mathbf{x}$ defines a canonical Hamiltonian structure which implies the following set of Poisson brackets for the holomorphic components:

$$\{x_R(\tau), x_R(\tau')\} = \{y_R(\tau), y_R(\tau')\} = -\tfrac{1}{4}\epsilon(\tau - \tau'), \qquad \{x_R(\tau), y_R(\tau')\} = 0, \tag{38}$$

with

$$\epsilon(\tau) = 2l + 1 \quad \text{for} \quad 2\pi Rl < \tau < 2\pi R(l+1); \quad l \in \mathbb{Z}.$$

Using equations (38) it is easy to show that

$$\begin{aligned} &\{\psi(\tau), \psi(\tau')\} = \epsilon(\tau - \tau')\psi(\tau)\psi(\tau'), \\ &\{\psi^*(\tau), \psi^*(\tau')\} = \epsilon(\tau - \tau')\psi^*(\tau)\psi^*(\tau'), \\ &\{\psi^*(\tau), \psi(\tau')\} = \delta'(\tau - \tau') - \epsilon(\tau - \tau')\psi^*(\tau)\psi(\tau'). \end{aligned} \tag{39}$$

The Poisson bracket algebra (39) is well known to describe the second Hamiltonian structure for the AKNS soliton hierarchy [16, 17] and equations (34) can be interpreted as a transform to the Dorboux variables (38) for this Hamiltonian structure.

The fields (37) are local densities of the first AKNS Hamiltonians:

$$I_s^{(\mathrm{class})} = \mathrm{i}^{1-s}\int_0^{2\pi R} \mathrm{d}\tau\, w_{s+1}(\tau). \tag{40}$$

In particular $I_2^{(\mathrm{class})}$ coincides with the Hamiltonian of the famous nonlinear Schrödinger equation [9].[5] There are an infinite number of IM in the form (40) which form a commutative Poisson bracket subalgebra

$$\left\{I_s^{(\mathrm{class})}, I_{s'}^{(\mathrm{class})}\right\} = 0. \tag{41}$$

[5] The classical IM in equation (40) are normalized in accordance with the convention from the book [17]. If (x_R, y_R) in (34) are real functions of the real variable τ, then ψ and ψ^* is a complex conjugated pair and the corresponding nonlinear Schrödinger equation (42) is in the repulsive regime.

It is remarkable that all AKNS local densities w_s (40) satisfy equation (31). This statement is essentially known in the literature in the context of Bäcklund transformation (see, e.g., appendix B in [18]), even though it is not formulated in the language of the perturbed boundary theory.

It is well known (see, e.g., [17]) that the AKNS flows,

$$\frac{\partial\psi}{\partial t_s} = \big\{I_s^{(\mathrm{class})}, \psi(t_1, t_2\ldots)\big\}, \qquad \frac{\partial\psi^*}{\partial t_s} = \big\{I_s^{(\mathrm{class})}, \psi^*(t_1, t_2\ldots)\big\}, \tag{42}$$

describe isospectral deformations of the first-order differential operator

$$\mathcal{L} = -\partial_\tau - \frac{\mathrm{i}\lambda}{R}H + \psi^* E + \psi F, \tag{43}$$

where ψ, ψ^* are the quasiperiodic fields on the segment $0 \leqslant \tau \leqslant 2\pi R$ (36) and E, F and H are the generators of the Lie algebra $sl(2)$,

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{44}$$

More precisely, if we define the $(2j+1)\times(2j+1)$ monodromy matrices $\mathbf{M}_j(\lambda)$, $j = 0, 1/2, 1, 3/2\ldots$, corresponding to the $(2j+1)$-dimensional representation π_j of (44), as

$$(\chi_1(\tau+2\pi R), \ldots \chi_{2j+1}(\tau+2\pi R)) = (\chi_1(\tau), \ldots \chi_{2j+1}(\tau))\mathbf{M}_j(\lambda), \tag{45}$$

where $\chi_k(\tau)$ are $(2j+1)$ linear independent solutions to the auxiliary linear problem

$$\pi_j[\mathcal{L}]\chi = 0, \tag{46}$$

then the transfer matrices

$$T_j(\lambda) = \mathrm{Tr}_{\pi_j}[\mathrm{e}^{-2\pi\mathrm{i}\xi_y H}\mathbf{M}_j(\lambda)] \tag{47}$$

are involutive (with respect the Poison structure (39)) IM of the AKNS flows:

$$\big\{T_j(\lambda), I_s^{(\mathrm{class})}\big\} = 0 \tag{48}$$

and

$$\{T_j(\lambda), T_{j'}(\mu)\} = 0. \tag{49}$$

As a matter of fact, the transfer matrices $T_j(\lambda)$ are not independent for different j and can be algebraically expressed in terms of $T(\lambda) \equiv T_{\frac{1}{2}}(\lambda)$ corresponding to the fundamental representation of $sl(2)$. The latter can be thought of as a generating function of the local IM (40) as it expands in the $\lambda \to \infty$ asymptotic series [17]:

$$T(\lambda) = 2\cosh(2\pi\nu(\lambda)), \tag{50}$$

with

$$\mathrm{i}\nu(\lambda) \simeq -\lambda - \xi_y + \frac{1}{2\pi}\sum_{s=1}^{\infty} I_s^{(\mathrm{class})}\left(\frac{R}{2\lambda}\right)^s \qquad \text{as} \quad \lambda \to \infty. \tag{51}$$

3. Semiclassical quantization

Here we study the semiclassical behaviour of the boundary amplitude (19) using the path integral approach. For the sake of discussion, it is convenient to consider the conformal map of the semi-infinite cylinder, $\tau \equiv \tau + 2\pi R$, $\sigma \geqslant 0$, to the disc of radius R:

$$\frac{z}{R} = \mathrm{e}^{-(\sigma+\mathrm{i}\tau)/R}, \qquad \frac{\bar{z}}{R} = \mathrm{e}^{-(\sigma-\mathrm{i}\tau)/R}. \tag{52}$$

Then the overlap ${}_{\text{pert}}\langle B|\mathbf{P}\rangle$ with the Fock vacuum $|\mathbf{P}\rangle$ relates to the unnormalized one-point function of associated primary field inserted at the centre of the disc,

$$\langle e^{i\mathbf{P}\cdot\mathbf{X}}(0,0)\rangle_{\text{disc}} = R^{1/3-\mathbf{P}^2/2}\,{}_{\text{pert}}\langle B|\mathbf{P}\rangle, \tag{53}$$

and $R^{1/3}{}_{\text{pert}}\langle B|\mathbf{0}\rangle$ is the disc partition function[6]. The one-point function (53) can be represented in terms of the functional integral as follows,

$$\langle e^{i\mathbf{P}\cdot\mathbf{X}}(0,0)\rangle_{\text{disc}} = \int \mathcal{D}X\mathcal{D}Y\, e^{iPX+iQY}(0,0)\, e^{-\mathscr{A}[\mathbf{X}]-\mathscr{A}_{\text{pert}}[\mathbf{X}_B]}, \tag{54}$$

where $\mathbf{P} = (P, Q)$, and the integration variables $X(z,\bar{z}), Y(z,\bar{z})$ are assumed to obey constraint (2) at the boundary $|z| = R$. Note that at the classical limit the effect of exponential insertion in (54) can be accounted for by imposing the asymptotic condition (23) in the cylindrical frame. For this reason the vector $\boldsymbol{\xi}$ in (23) is proportional to $\mathbf{P}$:

$$\boldsymbol{\xi} = \frac{\mathbf{P}}{2\sqrt{n}}. \tag{55}$$

When P and Q are pure imaginary, some insight can be gained by making a shift of integration variables,

$$\mathbf{X} \to \mathbf{X} + i\mathbf{P}\log\frac{|z|}{R}, \tag{56}$$

in the functional integral (54), which brings it to the form

$$\langle e^{i\mathbf{P}\cdot\mathbf{X}}(0,0)\rangle_{\text{disc}} = R^{-\mathbf{P}^2/2}\int \mathcal{D}X\mathcal{D}Y\, e^{-\mathscr{A}_{\text{bulk}}[\mathbf{X}]-\mathscr{A}_{\text{bound}}[\mathbf{X}_B]}, \tag{57}$$

where the boundary action is given by

$$\mathscr{A}_{\text{bound}} = -\oint_{|z|=R}\frac{dz}{2\pi z}(PX_B + QY_B + iRU(X_B, Y_B))(z). \tag{58}$$

In this section we evaluate leading semiclassical contribution to the overlap (19) by direct calculation of the functional integral (57) in the saddle-point approximation. This will give some intuition about its structure.

3.1. 'Light' vertex insertion

Making an attempt to calculate (54) we immediately encounter a problem. The potential (30) is unbounded as $y \to \pm\frac{\pi}{2}$. In fact, in order to ensure a convergence of the functional integral we must place the overall pure imaginary constant C in (30):[7]

$$C = i\lambda. \tag{59}$$

For real positive λ, the path integral (57) seems to be well defined, but the pure imaginary boundary potential $U(\mathbf{X}_B)$ will certainly break down the unitarity of the underline QFT in the Minkowski coordinates $(\sigma, i\tau)$.

Let us assume that the parameters P, Q and λ are sufficiently small, i.e., the boundary action (58) has no appreciable effect on the saddle-point configuration. To be precise we write

$$(P, Q) = \frac{2}{\sqrt{n}}(p, q) \tag{60}$$

[6] We always assume that the exponential field in (53) is defined according to the usual CFT conventions [19]. The factor $R^{1/3}$ appears due to the conformal anomaly.

[7] It deserves to draw reader's attention that in (59) we use the same notation λ as for the spectral parameter in equation (43). The reason is discussed at the end of this section.

and assume that

$$p, q \qquad \text{and} \qquad n\lambda \sim 1 \qquad \text{as} \quad n \to \infty. \tag{61}$$

Note that these conditions are somewhat different than those considered in the previous section, where the limit $n \to \infty$ was taken under the assumption that $\xi_x = p/n$, $\xi_y = q/n$ and λ are fixed.

With the condition (61) the action is minimized by the trivial classical solutions $\mathbf{X}(z, \bar{z}) = \mathbf{X}_0$, where $\mathbf{X}_0 = (X_0, Y_0)$ is an arbitrary point on the hairpin (2). Therefore

$$Z_{\text{class}} = \int_{\text{hairpin}} \mathrm{d}\mathcal{M}(\mathbf{X}_0)\, \mathrm{e}^{2\mathrm{i}(pX_0+qY_0)/\sqrt{n}}\, \mathrm{e}^{-RU(\mathbf{X}_0)}, \tag{62}$$

where the integration measure $\mathrm{d}\mathcal{M}(\mathbf{X}_0)$ is determined by integrating out fluctuations around the classical solution in the Gaussian approximation. Of course, there is no need of actually evaluating this Gaussian functional integral to figure out the answer. If one writes $\mathbf{X}(z, \bar{z}) = \mathbf{X}_0 + \delta\mathbf{X}(z, \bar{z})$, and splits the fluctuational part $\delta\mathbf{X}$ into the components normal and tangent to the hairpin at the point $\mathbf{X}_0$, the components are to satisfy the Dirichlet and the Neumann boundary conditions, respectively. Therefore one just has to take the product of the known (see, e.g., appendix to [20]) disc partition functions with these two boundary conditions. As the result, $\mathrm{d}\mathcal{M}(\mathbf{X}_0) = \frac{g_D^2}{2\pi}\, \mathrm{d}\ell(\mathbf{X}_0)$, where $\mathrm{d}\ell(\mathbf{X}_0) = \sqrt{(\mathrm{d}X_0)^2 + (\mathrm{d}Y_0)^2}$ is the length measure of the hairpin, and $g_D = 2^{-\frac{1}{4}}$ is the 'boundary degeneracy' [21] associated with the Dirichlet conformal boundary condition[8].

Certainly the above analysis ignores the presence of (unbounded) boundary potential. Its proper account leads to the one-loop renormalization of the coupling constant λ. Indeed, the form of the boundary potential in the classical limit suggests that

$$U(\mathbf{X}_B) \to \pm\mathrm{i}\frac{n\lambda}{R}\, \mathrm{e}^{-\frac{X_B}{\sqrt{n}}} \qquad \text{as} \quad X_B \to -\infty, \tag{63}$$

where the sign factor is dictated by the choice of hairpin asymptote $Y_B = \mp\frac{\pi}{2\sqrt{n+2}}$. The anomalous dimension of the boundary operator $\mathrm{e}^{-\frac{X_B}{\sqrt{n}}}$ with respect to the hairpin stress–energy tensor (3) is given by

$$d_{\text{pert}} = -\frac{2}{n}. \tag{64}$$

If one imposes now the normalization condition

$$\left[\frac{\partial}{\partial E_*}\langle X_B\rangle_{\text{disc}}\right]_{E_*=0} = 0 \tag{65}$$

on the expectation value of renormalized boundary field X_B, then the renormalized coupling λ is related to the RG invariant scale E_* as in equation (16).

Although equation (16) is a result of the semiclassical consideration, in the next section we shall bring forward arguments which show that it is perturbatively exact, i.e., a renormalization scheme exists in which it is exact to all orders in the $\frac{1}{n}$-expansion.

All the above add up to the following integral representation for the semiclassical boundary amplitude:

$$Z_{\text{class}} = g_D^2\sqrt{n}\left(\tfrac{n}{2}r^2\lambda\right)^{-\mathrm{i}p} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}y}{2\pi}\, (\cos(y))^{2\mathrm{i}p-1}\, \mathrm{e}^{2\mathrm{i}qy}\, \mathrm{e}^{\mathrm{i}n\lambda\tan(y)}, \tag{66}$$

[8] The definition is as follows: $g_D = \langle B_D|\mathbf{P}\rangle$, where $|B_D\rangle$ is the boundary state of *uncompactified* boson X with the Dirichlet boundary condition $X_B = 0$, and the primary states $|P\rangle$ are delta-normalized, $\langle P \mid P'\rangle = \delta(P - P')$.

where the additional factor $\left(\frac{n}{2}r^2\lambda\right)^{-\mathrm{i}p}$ containing an arbitrary constant r is introduced to ensure the normalization condition (65). This factor can be also understood as an artefact of an additive counterterm for the bare boundary field X_B. Integral (66) is expressed (see equation (13), section 6.11.2 in [23]) in terms of Kummer's solution $U(a,b;z)$ of the confluent hypergeometric equation [24]:

$$Z_{\mathrm{class}} = g_D^2 r^{-2\mathrm{i}p} \frac{\sqrt{n}(2n\lambda)^{-\mathrm{i}p}}{\Gamma\left(\frac{1}{2}-q+\mathrm{i}p\right)} \mathrm{e}^{-n\lambda} U\left(\frac{1}{2}-\mathrm{i}p+q, 1-2\mathrm{i}p; 2n\lambda\right). \qquad (67)$$

Here we assume that $\lambda > 0$ and choose the principal brunch of the multivalued function $U(a,b;z)$ such that for positive z

$$U(a,b;z) \to z^{-a} \quad \text{as} \quad z \to +\infty. \qquad (68)$$

It is also instructive to rewrite the semiclassical boundary amplitude (67) in terms of the hypergeometric series $M(a,b,z) = 1 + \frac{az}{b} + \frac{a(a+1)}{2!b(b+1)}z^2 + \cdots$:

$$Z_{\mathrm{class}} = Z^{(-)}_{\mathrm{class}}(p,q)F^{(-)}_{\mathrm{class}}(p,q|\lambda) + Z^{(+)}_{\mathrm{class}}(p,q)F^{(+)}_{\mathrm{class}}(p,q|\lambda), \qquad (69)$$

where

$$\begin{aligned} Z^{(-)}_{\mathrm{class}}(p,q) &= g_D^2 r^{-2\mathrm{i}p} \frac{\sqrt{n}(2n\lambda)^{-\mathrm{i}p}\Gamma(2\mathrm{i}p)}{\Gamma\left(\frac{1}{2}+q+\mathrm{i}p\right)\Gamma\left(\frac{1}{2}-q+\mathrm{i}p\right)}, \\ Z^{(+)}_{\mathrm{class}}(p,q) &= g_D^2 r^{-2\mathrm{i}p} \frac{\sqrt{n}}{\pi}(2n\lambda)^{\mathrm{i}p}\cosh(\pi(p+\mathrm{i}q))\Gamma(-2\mathrm{i}p), \end{aligned} \qquad (70)$$

and

$$F^{(\mp)}_{\mathrm{class}}(p,q|\lambda) = M\left(\tfrac{1}{2} \mp \mathrm{i}p + q, 1 \mp 2\mathrm{i}p; 2n\lambda\right). \qquad (71)$$

In this form the nature of singular behaviour as $\lambda \to 0$ is more explicit. Let us make a (trivial) observation that the poles of $Z^{(-)}_{\mathrm{class}}(p,q)$ in the variable p in the first term in (69) are cancelled by the poles in the higher terms of the confluent hypergeometric series $F^{(+)}_{\mathrm{class}}$ in the second term, and vice versa, it makes the full partition function an entire function of parameters p and q.

3.2. 'Heavy' vertex insertion

Although the above result (69) was derived under the assumption that P, Q and λ are small, it needs little fixing to become valid for much larger values of these parameters. When $(P,Q) = 2\sqrt{n}(\xi_x, \xi_y)$ become as large as $\sqrt{n}$,

$$\xi_x, \xi_y \qquad \text{and} \qquad \lambda \sim 1 \qquad \text{as} \quad n \to \infty, \qquad (72)$$

the vertex insertion in (54) and the boundary potential is 'heavy', i.e., it must be treated as a part of the action, and they affect the saddle-point analysis. The saddle-point configuration(s) is still a constant field, but now $\mathbf{X}_0$ is not an arbitrary point on the curve (2), but has to extremize the boundary action (58)

$$\mathscr{A}_{\mathrm{bound}}[\mathbf{X}_0] = -\mathrm{i}\mathbf{P}\cdot\mathbf{X}_0 + RU(\mathbf{X}_0). \qquad (73)$$

There are two solutions of the saddle-point equation which are real valued for real values of the parameter

$$\nu_0 = \sqrt{\xi_x^2 - 2\lambda\xi_y - \lambda^2}. \qquad (74)$$

One of them corresponds to the minimum, and another to the maximum of $\mathrm{i}\mathscr{A}_{\mathrm{bound}}[\mathbf{X}_0]$. The saddle-point action produces nothing but the $p, q \to \infty$ asymptotic form of the expression

(69)—after all, this asymptotic of the integral (66) is controlled by the same saddle points. One can observe that if one splits the constant-mode integration into two parts, as was suggested in (69), the parts receive contributions from different saddle points—one from the 'minimum' and one from the 'maximum' one. What makes the difference at $(P, Q) \sim \sqrt{n}$ is the proper treatment of non-constant modes. One writes

$$\mathbf{X}(z, \bar{z}) = \mathbf{X}_* + \mathbf{t}_* \delta X_t(z, \bar{z}) + \mathbf{n}_* \delta X_n(z, \bar{z}), \tag{75}$$

where $\mathbf{X}_*$ is the position of the saddle point on the hairpin, and $\mathbf{t}_*$ and $\mathbf{n}_*$ are unit vectors tangent and normal to the hairpin at this point. Then for small δX_t the boundary constraint (2) reads

$$\delta X_n = -\mathrm{e}^{\frac{x_0}{\sqrt{n}}} \frac{\delta X_t^2}{2\sqrt{n}} + O\big(\delta X_t^3\big), \tag{76}$$

and, up to higher-order terms, the boundary action (58) can be written as

$$\mathcal{A}_B = A_B[\mathbf{X}_*] \mp \nu_0 \oint \frac{\mathrm{d}z}{2\pi z} \delta X_t^2, \tag{77}$$

with the coefficient $\nu_0 > 0$ being given by (74); then the sign minus (plus) in (77) applies to the 'minimum' ('maximum') saddle point. Thus, while to the leading approximation the normal component δX_n still can be treated with the Dirichlet boundary condition, the 'boundary mass' term in (77) has to be taken into account in evaluating the contribution from δX_t. Using the well-known boundary amplitude of the free field with quadratic boundary interaction [22], one finds that equation (69) would apply to the case of $(P, Q) \sim \sqrt{n}$ as well if one puts corresponding additional factors in the two terms in (69), i.e., replaces $Z^{(\mp)}_{\text{class}}(p, q)$ there by

$$\tilde{Z}^{(\mp)}_{\text{class}}(P, Q) = Z^{(\mp)}_{\text{class}}\left(\frac{\sqrt{n}}{2} P, \frac{\sqrt{n}}{2} Q\right) \Gamma\left(1 \pm \mathrm{i}\sqrt{\frac{P^2}{n} - \frac{4Q}{\sqrt{n}}\lambda - 4\lambda^2}\right), \tag{78}$$

of course in this case one can use the $p, q \to \infty$ asymptotic forms of the factors (70) and (71). More explicitly,

$$Z_{\text{class}} \sim \frac{g_D^2}{\sqrt{2\pi}} \left(\frac{\Gamma(1 + 2\mathrm{i}\nu_0)}{\sqrt{2\mathrm{i}\nu_0}} \mathrm{e}^{\mathrm{i}nS_-} + \frac{\Gamma(1 - 2\mathrm{i}\nu_0)}{\sqrt{-2\mathrm{i}\nu_0}} \mathrm{e}^{\mathrm{i}nS_+}\right), \tag{79}$$

with

$$\lambda \frac{\partial S_\pm}{\partial \lambda} = \pm \nu_0. \tag{80}$$

Here the two terms in (79) correspond to the different saddle points of the boundary potential (73). The structure (79) is common for the semiclassical form of Baxter's Q-function (see [25]). It is instructive to note in this connection that the classical transfer matrix $T^{(\text{vac})}(\lambda)$ corresponding to simplest ('vacuum') choice $\mathbf{x}_R = \frac{\mathrm{i}\xi}{R}(\sigma + \mathrm{i}\tau)$ in equations (34), (43), can be written in the form (50): $T^{(\text{vac})}(\lambda) = 2\cosh(2\pi\nu_0)$, where ν_0 is the same function (74) as in equation (80). One can easily recognize in these equations the classical counterpart of the Baxter relation (18).

4. Integrability of the IPH model

In this section we would like to argue that the quantum IPH model is integrable. Our first step towards the consistent quantum theory is the quantization of the local IM of AKNS soliton hierarchy.

4.1. Quantum local IM

Let us first recall structure of the hairpin W-algebra [1].[9] Its generating currents $W_s(u)(u = \sigma + \mathrm{i}\tau)$ have spins $s = 2, 3, 4, \ldots$, and can be characterized by the condition that they commute with two 'screening operators (charges)', i.e.,

$$\oint_u \mathrm{d}v\, W_s(u)\mathcal{V}_\pm(v) = 0, \tag{81}$$

where

$$\mathcal{V}_\pm = \mathrm{e}^{\sqrt{n}X_R \pm \mathrm{i}\sqrt{n+2}Y_R} \tag{82}$$

and X_R, Y_R in the exponential stand for the holomorphic parts of the corresponding local fields. The integration in equation (81) is over a small contour around the point u; vanishing of integral (81) implies that the singular part of the operator product expansion of $W_s(u)\mathcal{V}_\pm(v)$ is a total derivative $\partial_v(\cdots)$. This condition fixes $W_s(u)$ uniquely up to normalization and adding derivatives and composites of the lower-spin W-currents. For instance, the first holomorphic current beyond (3) can be written as (4) where the ambiguity in adding a term proportional to ∂W_2 is fixed by demanding that (4) is a conformal primary. The higher currents $W_4, W_5, \ldots$ can be found either by a direct computation of the operator product expansions with the screening exponentials (82), or recursively, by studying the singular parts of the operator product expansions of the lower currents, starting with $W_3(u)W_3(v)$ and continuing upward. Thus, the product $W_3(u)W_3(v)$ contains singular term $\sim (u-v)^{-2}$ which involves, besides the derivatives $\partial^2 W_2$ and the composite operator W_2^2, the new current W_4. Further operator products with W_4 define higher W's, etc. In this sense the W-algebra is generated by the two basic currents W_2 and W_3.

As the matter of fact, there is the third, the most effective way to generate all W-currents. It is based on the observation that the screening operators associated with the exponentials (82) commute with the parafermionic currents

$$\begin{aligned}\Psi(u) &= \mathrm{i}(\sqrt{n+2}\partial Y + \mathrm{i}\sqrt{n}\partial X)\,\mathrm{e}^{\frac{2\mathrm{i}}{\sqrt{n+2}}Y_R(u)},\\ \Psi^*(u) &= \mathrm{i}(\sqrt{n+2}\partial Y - \mathrm{i}\sqrt{n}\partial X)\,\mathrm{e}^{-\frac{2\mathrm{i}}{\sqrt{n+2}}Y_R(u)},\end{aligned} \tag{83}$$

in the same sense as the W-currents do, i.e.,

$$\oint_u \mathrm{d}v\ \Psi(u)\mathcal{V}_\pm(v) = 0, \qquad \oint_u \mathrm{d}v\ \Psi^*(u)\mathcal{V}_\pm(v) = 0. \tag{84}$$

The fields (83) extend the notion of the $\mathbb{Z}_k$ parafermions of [28] to non-integer $k = -n-2$. Clearly, they can be also treated as a quantum version of the classical nonlocal currents (34). In spite of the fact that the fields (84) are not local, both Ψ and Ψ^* are local with respect to the exponentials (82); hence the integration contour in (84)—a small contour around u—is indeed a closed one. It follows from (84) that all the fields generated by the operator product expansion of $\Psi(u)\Psi^*(v)$ satisfy equations (81) [6, 27]. Thus we have

$$\begin{aligned}\Psi(u)\Psi^*(v) = (u-v)^{-\frac{2}{n+2}}\bigg\{&\frac{n+2}{(u-v)^2} + \frac{n}{2}\big(W_2(u) + W_2(v)\big)\\ &- \frac{\mathrm{i}(u-v)}{2\sqrt{n+2}}(W_3(u) + W_3(v)) + \cdots\bigg\},\end{aligned} \tag{85}$$

where W_2 and W_3 are the same as in (3) and (4), and the higher-order terms involve the higher-spin W-currents.

[9] This W-algebra was known for a long while (see [6, 26, 27]).

After the brief review of the hairpin extended conformal symmetry, we turn to the description of the quantum AKNS local integrals which constitute the Abelian subalgebra of the W-algebra. The main idea is based on examination of the asymptotic form of classical boundary potential (63). It suggests defining the quantum local IM $\mathbb{I}_s$ as elements of the hairpin W-algebra in the form (8) commuting with the additional screening charge, i.e.,

$$\oint_u \mathrm{d}v\, P_{s+1}(u)\mathcal{V}_0(v) = \partial_u(\cdots), \tag{86}$$

where

$$\mathcal{V}_0 = \mathrm{e}^{-\frac{2X_R}{\sqrt{n}}}. \tag{87}$$

Remarkably the condition (86) indeed defines, up to an overall multiplicative normalization, a local integral of motion $\mathbb{I}_s$ for each $s = 1, 2, 3, \ldots$. In particular, one can show that $P_2 = W_2$, $P_3 = \mathrm{i}W_3$ and

$$\begin{aligned} P_4 = n(\partial X)^4 + 6n(\partial X)^2(\partial Y)^2 + (5n+4)(\partial Y)^4 \\ + 6(n+1)\sqrt{n}\partial^2 X(\partial Y)^2(n^2+3n+1)(\partial^2 X)^2 + (n^2+4n+2)(\partial^2 Y)^2, \end{aligned} \tag{88}$$

where we disregard all terms which are total derivatives and do not contribute to the $\mathbb{I}_3$. Also it is straightforward to check that the such defined $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ mutually commute. One may also note that in the classical limit P_4 (88) turns out to be the classical local density $(-w_4)$ (33) up to a total derivative: $P_4 \to n^3(-w_4 + \partial(\ldots))$ as $n \to \infty$.

It seems likely that the formal proof of existence and uniqueness solution of (86) for $s = 5, 6, \ldots$ can be obtained along the line of [29–31]. Currently it is not available. Nevertheless, we take the foregoing facts as a strong indication that an infinite sequence of commuting IM $\{\mathbb{I}_s\}_{s=1}^{\infty}$ in the form (8) exists, whose first local densities are given by equations (9), (11) and (88). It is expected that the operators $\mathbb{I}_s$ become the AKNS Hamiltonians (40) in the classical limit.

4.2. Diagonalization of local IM

It seems natural to take an attitude that the boundary state of quantum IPH model satisfies the integrability condition (7) for all the local IM from quantum AKNS series. Thus we run into the problem of simultaneous diagonalization of the AKNS local IM in the Fock space $\mathcal{F}_{\mathbf{P}}$.

Let us consider the normal mode expansion of the quantum holomorphic current $\partial\mathbf{X}$:

$$\partial\mathbf{X} = \frac{\mathrm{i}}{R}\left(\frac{1}{2}\mathbf{P} + \sum_{k\neq 0}\mathbf{X}_k\,\mathrm{e}^{\frac{k(\sigma+\mathrm{i}\tau)}{R}}\right). \tag{89}$$

Here $\mathbf{P} = (P, Q)$ is the zero-mode momentum and $X_k^{\mu} = (X_k, Y_k)$ are the oscillatory modes. The canonical quantization procedure of the Poisson bracket algebra (38) leads to the following set of commutation relations,

$$\left[X_k^{\mu}, X_s^{\nu}\right] = \frac{k}{2}\delta^{\mu\nu}\delta_{k+s,0}. \tag{90}$$

In appendix A we present explicit expressions for the first local IM in terms of the oscillatory modes X_k^{μ}. As it follows from equations (A.2), (A.3), $\mathbb{I}_s$ are well-defined operators acting in the Fock space $\mathcal{F}_{\mathbf{P}}$. The later is the highest weight module over the Heisenberg algebra (90) with the highest vector $|0, \mathbf{P}\rangle$ ('vacuum') defined by the equation

$$X_k^{\mu}|0, \mathbf{P}\rangle = 0, \qquad k = 1, 2, 3, \ldots. \tag{91}$$

The zero-mode momenta, P and Q, act in the given Fock space as c-numbers. The space $\mathcal{F}_{\mathbf{P}}$ naturally splits into the sum of finite-dimensional 'level subspaces'

$$\mathcal{F}_{\mathbf{P}} = \oplus_{\ell=0}^{\infty} \mathcal{F}_{\mathbf{P}}^{(\ell)}, \qquad \mathbb{L}\mathcal{F}_{\mathbf{P}}^{(\ell)} = \ell \mathcal{F}_{\mathbf{P}}^{(\ell)}. \tag{92}$$

Since the grading operator $\mathbb{L}$ essentially coincides with $\mathbb{I}_1$,

$$\mathbb{I}_1 = R^{-1}\left(\mathbb{L} + \frac{\mathbf{P}^2}{4} - \frac{1}{12}\right), \tag{93}$$

all the local IM act invariantly in the level subspaces $\mathcal{F}_{\mathbf{P}}^{(\ell)}$. Therefore diagonalization of $\mathbb{I}_s$ in a given level subspace reduces to a finite algebraic problem which however rapidly becomes very complex for higher levels. Note that for $n \geqslant 0$ and real P, Q, the operators $\mathbb{I}_s$ are Hermitian with respect to the canonical conjugation $\left(X_k^{\mu}\right)^{\dagger} = X_{-k}^{\mu}$, so their spectra are real and their eigenvectors are orthogonal to each other. Here we list only the vacuum eigenvalues for the first few $\mathbb{I}_s$:

$$\mathbb{I}_s|0, \mathbf{P}\rangle = R^{-s} I_s^{(\text{vac})}(P, Q)|0, \mathbf{P}\rangle, \tag{94}$$

where

$$\begin{aligned}
I_1^{(\text{vac})} &= \frac{P^2}{4} + \frac{Q^2}{4} - \frac{1}{12},\\
I_2^{(\text{vac})} &= \frac{Q}{12}((3n+2)Q^2 + 3nP^2 - 2n - 1),\\
I_3^{(\text{vac})} &= \frac{n}{16}\left(P^4 - P^2 + \frac{1}{12}\right) + \frac{3n}{8}\left(P^2 - \frac{1}{6}\right)\left(Q^2 - \frac{1}{6}\right)\\
&\quad + (5n+4)\left(Q^4 - Q^2 + \frac{1}{12}\right) + \frac{(n+3)(2n+1)}{240}.
\end{aligned} \tag{95}$$

An explicit form of the eigenvalues and the corresponding eigenvectors in $\mathcal{F}_{\mathbf{P}}^{(\ell)}$ is somewhat cumbersome even for small ℓ. For this reason we do not present it here. However, it suggests that even the first three local IM resolve all degeneracies in $\mathcal{F}_{\mathbf{P}}$, i.e., they orthonormalized eigenvectors $\{|\alpha, \mathbf{P}\rangle\}$, which are labelled by some index α, form a basis in the Fock space. This is in turn leads to the structure (13). Note that the amplitude $B_0(\mathbf{P})$ in equation (13) corresponding to the normalized highest vector $|0, \mathbf{P}\rangle \in \mathcal{F}_{\mathbf{P}}$,

$$\langle \mathbf{P}', 0|0, \mathbf{P}\rangle = \delta(\mathbf{P}' - \mathbf{P}), \tag{96}$$

coincides with the complex conjugated overlap Z^* (19).

5. Dual form of the IPH model

Here we discuss an alternative description of the IPH model and explore the short distance behaviour of the boundary amplitude (19) beyond the semiclassical approximation.

5.1. Dual Hamiltonian

In the recent work [32], it was suggested that the hairpin model admits equivalent 'dual' description. The dual model involves a two-component Bose field $\left(X(\sigma,\tau), \tilde{Y}(\sigma,\tau)\right)$ (where $\tilde{Y}$ is interpreted as the T-dual of Y [10]) on the semi-infinite cylinder, which has the free-field

[10] The T-dual of free massless field is defined as usual, through the relations

$$Y = Y_R(\sigma + \mathrm{i}\tau) + Y_L(\sigma - \mathrm{i}\tau), \qquad \tilde{Y} = Y_R(\sigma + \mathrm{i}\tau) - Y_L(\sigma - \mathrm{i}\tau).$$

dynamics in the bulk, and obeys *no constraint* at the boundary $\sigma = 0$; instead it interacts with an additional 'boundary' degree of freedom. In the dual description it is convenient (but not necessary) to use the Hamiltonian picture where the cyclic coordinate $\tau \equiv \tau + 2\pi R$ is treated as the Matsubara (compact Euclidean) time τ. In this picture the boundary amplitude $Z^{(-)}(P, Q) = {}_{\rm hair}\langle B|\mathbf{P}\rangle$ admits the dual representation as the trace

$$Z^{(-)}(P, Q) = \mathrm{Tr}_{\tilde{\mathcal{H}}}[\mathrm{e}^{-2\pi R \hat{H}_{\rm hair}}], \tag{97}$$

taken over the space $\tilde{\mathcal{H}} = \mathcal{H}_{X,\tilde{Y}} \otimes \mathbb{C}^2$, where $\mathcal{H}_{X,\tilde{Y}}$ is the space of states of the two-component boson $(X(\sigma), \tilde{Y}(\sigma))$ on the half-line $\sigma \geqslant 0$ (with no constraint at $\sigma = 0$) and $\mathbb{C}^2$ is the two-dimensional space representing the new boundary degree of freedom. The dual Hamiltonian in (97) consists of the bulk and the boundary parts,

$$\hat{H}_{\rm hair} = \hat{H}_{\rm bulk} - \frac{\mathrm{i}}{2\pi R} P X_B - \frac{\mathrm{i}\sqrt{n+2}}{4R} Q\sigma_3 + \mu_-\big[\sigma_+ \,\mathrm{e}^{\frac{\sqrt{n}}{2}X_B + \mathrm{i}\frac{\sqrt{n+2}}{2}\tilde{Y}_B} + \sigma_- \,\mathrm{e}^{\frac{\sqrt{n}}{2}X_B - \mathrm{i}\frac{\sqrt{n+2}}{2}\tilde{Y}_B}\big]. \tag{98}$$

The bulk part, $\hat{H}_{\rm bulk}$, is a 'free' Hamiltonian corresponding to the bulk action (1). The boundary term describes coupling of the boundary values $(X_B, \tilde{Y}_B) \equiv (X, \tilde{Y})|_{\sigma=0}$ of the field operators to the additional boundary degree of freedom represented by $\mathbb{C}^2$ ($\sigma_\pm$ and σ_3 are the Pauli matrices acting in $\mathbb{C}^2$). We also include in equation (98) an extra parameter $\mu_- > 0$. In the unperturbed hairpin model μ_- can be always eliminated by shifting of the field X, but this coupling will be of considerable importance for the IPH model.

We emphasize the similarity of the boundary vertex operators in equation (98) to the screening charges (82) determining the extended conformal symmetry of the hairpin model. To construct the dual Hamiltonian for the perturbed theory, it is crucial to observe that the local IM $\mathbb{I}_s$ defined through the condition (86), also commute with the additional 'dual' screening charge:

$$\oint_u \mathrm{d}v\, P_{s+1}(u)\mathcal{V}_0^{(\rm dual)}(v) = \partial_u(\cdots), \tag{99}$$

where

$$\mathcal{V}_0^{(\rm dual)} = \mathrm{e}^{-2\sqrt{n}X_R}. \tag{100}$$

For $s = 1, 2, 3$ equation (99) was checked by the direct calculation using the explicit forms of $\mathbb{I}_s$. We take this as a strong indication that it holds for all the local IM from AKNS series. This in turn suggests a possibility of dual description of the IPH model by means of the Hamiltonian $\hat{H}_{\rm iph} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$,

$$\hat{H}_{\rm iph} = \hat{H}_{\rm hair} + \Sigma\, \mathrm{e}^{-\sqrt{n}X_B}, \tag{101}$$

where Σ is some 2×2 matrix acting in the $\mathbb{C}^2$-component of the Hilbert space $\tilde{\mathcal{H}}$. In what follows we will argue that the matrix Σ has the diagonal form,

$$\Sigma = \mu_+ \,\mathrm{e}^{-\frac{\pi \mathrm{i} n}{2}\sigma_3}, \tag{102}$$

where μ_+ is an arbitrary constant. Note that the Hamiltonian $\hat{H}_{\rm iph}$ (101), (102) is not Hermitian. This is consistent with our previous observation from section 3 that the IPH model, once considered in the Minkowski space, is a nonunitary QFT.

5.2. Short distance expansion

To justify the conjectured form of dual Hamiltonian we shall study the short distance behaviour of the partition function corresponding to (101)

$$Z = \mathrm{Tr}_{\tilde{\mathcal{H}}}[\mathrm{e}^{-2\pi R \hat{H}_{\rm iph}}]. \tag{103}$$

Qualitatively, one may expect that if $\mathrm{Im}\,P$ is not too small, the limit $R \to 0$ of the partition function Z is controlled by either the hairpin boundary operators $\mathrm{e}^{\frac{\sqrt{n}}{2}X_B \pm \mathrm{i}\frac{\sqrt{n+2}}{2}Y_B}$ (98) or the Liouville boundary vertex (101) depending on the sign $\mathrm{Im}\,P$, with some crossover at small $\mathrm{Im}\,P$.

More precisely, let us assume first that $\mathrm{Im}\,P < 0$. Then we can treat the second term in (101) as a perturbation and in the leading approximation

$$Z|_{R\to 0} \to Z^{(-)}(P, Q) \qquad (\mathrm{Im}\,P < 0), \tag{104}$$

where $Z^{(-)}(P, Q) = {}_{\mathrm{hair}}\langle B|\mathbf{P}\rangle$ is the hairpin boundary amplitude (97) found in the paper [1]:

$$Z^{(-)} = g_D^2 \left(2\pi R \frac{\mu_-^2}{n}\right)^{-\frac{2\mathrm{i}p}{n}} \frac{\sqrt{n}\,\Gamma(2\mathrm{i}p)\Gamma\left(1 + \frac{2\mathrm{i}p}{n}\right)}{\Gamma\left(\frac{1}{2} + q + \mathrm{i}p\right)\Gamma\left(\frac{1}{2} - q + \mathrm{i}p\right)}. \tag{105}$$

Here and below we often use the notations

$$p = \tfrac{1}{2}\sqrt{n}P, \qquad q = \tfrac{1}{2}\sqrt{n+2}Q. \tag{106}$$

Note that in the semiclassical case $n \gg 1$ the parameters p, q (106) are the same as in (60). One should not distinguish between n and $n + 2$ at the perturbative order discussed in section 3.

Now we turn to the case $\mathrm{Im}\,P > 0$, where the leading short distance behaviour is controlled by the Liouville vertex $\mathrm{e}^{-\sqrt{n}X_B}$. In fact, because of the matrix factor Σ (102), there are two noninteracting boundary Liouville field theories with the boundary 'cosmological constants' $\mu_+\,\mathrm{e}^{-\frac{\pi\mathrm{i}n}{2}}$ and $\mu_+\,\mathrm{e}^{\frac{\pi\mathrm{i}n}{2}}$. Using the result from the works [33, 34] one has

$$Z|_{R\to 0} \to Z^{(+)}(P, Q) \qquad (\mathrm{Im}\,P > 0), \tag{107}$$

with

$$Z^{(+)} = g_D^2 \frac{\sqrt{n}}{2\pi}\Gamma(-2\mathrm{i}p)\Gamma\left(1 - 2\mathrm{i}\frac{p}{n}\right)\left(\frac{2\pi\mu_+ R^{n+1}}{\Gamma(1-n)}\right)^{\frac{2\mathrm{i}p}{n}} (\mathrm{e}^{\mathrm{i}\pi q + \pi p} + \mathrm{e}^{-\mathrm{i}\pi q - \pi p}).$$

Here the last factor comes from the trace over $\mathbb{C}^2$-component of the Hilbert space $\tilde{\mathcal{H}}$.

Since $Z^{(\pm)}(P, Q)$ vanishes in the limit $R \to 0$ if P is taken in the 'wrong' half-plane (note the factors $R^{-\mathrm{i}\frac{2p}{n}}$ and $R^{2\mathrm{i}p\frac{(n+1)}{n}}$ in (105), (107)), this in turn suggests that the overall $R \to 0$ asymptotic of the partition function is correctly expressed by the sum

$$Z|_{R\to 0} \to Z^{(-)}(P, Q) + Z^{(+)}(P, Q). \tag{108}$$

Remarkably that equation (108) is in a perfect agreement with our semiclassical result (69), (78) provided

$$\frac{\mu_-^2}{\mu_+} = \frac{n}{\Gamma(1-n)}(r^2 R)^n, \tag{109}$$

and

$$\mu_-^2\mu_+ = \frac{\Gamma(1-n)}{16\pi^2 n}(2E_*)^{n+2}, \tag{110}$$

where E_* is the RG invariant 'physical scale' in the IPH model (16). It gives a strong support to the conjectured Hamiltonian (101).

What can be said about corrections to the leading asymptotic? Again we assume that $\mathrm{Im}\,P < 0$ and consider the perturbative effect of the second term in (101) to the hairpin boundary amplitude (104). In the unperturbed hairpin theory the parameter μ_- is a dimensionless constant. Let us eliminate μ_- from the hairpin Hamiltonian by shifting of the field X. Then the coupling μ_+ in front of the Liouville term in equations (101), (102) is replaced by $\mu_-^2\mu_+$. Since the anomalous dimension of the boundary operator $\mathrm{e}^{-\sqrt{n}X_B}$ with respect to the hairpin stress–energy tensor (3) is given by

$$d_{\text{dual}} = -n - 1, \tag{111}$$

then $\mu_-^2\mu_+ \sim E_*^{n+2}$, which is in an agreement with equation (110). Hence we deduce that the perturbative corrections to the leading asymptotic (104) should be in a form of power series expansion of the dimensionless parameter κ^{n+2}, with

$$\kappa = E_* R. \tag{112}$$

The case Im $P > 0$ can be analysed similarly and one comes to the same conclusion about the form of perturbative corrections to the leading asymptotic (107).

The power series expansion in κ^{n+2} becomes invisible in the limit $n \to \infty$, while the consideration from section 3 suggests that the short distance expansion should be in a form of power series of the dimensionless parameter $\lambda \sim \kappa^{\frac{n+2}{n}}$ (16). This structure is neatly captured by the form

$$Z = Z^{(-)}(P,Q)F^{(-)}(P,Q|\kappa) + Z^{(+)}(P,Q)F^{(+)}(P,Q|\kappa), \tag{113}$$

where the functions $F^{(\pm)}(P,Q|\kappa)$ are double power series in integer powers of κ^{n+2} and $\kappa^{\frac{n+2}{n}}$:

$$F^{(\pm)}(P,Q|\kappa) = \sum_{i,j=0}^{\infty} f_{i,j}^{(\pm)}(P,Q)\kappa^{i(n+2)+j\frac{(n+2)}{n}}, \tag{114}$$

with $f_{0,0} = 1$. The exact splitting into two terms in (113) is not easy to justify on general grounds, but can be supported by the following arguments.

First, recall that such splitting in the semiclassical expression (69) corresponds to isolating contributions from two saddle points. More importantly, the full expression has to take care of the poles of the factors $Z^{(\pm)}(P,Q)$—one should expect that the partition function (113) is an entire function of P. Expression (113) is the simplest form fit for this job. The poles of $Z^{(+)}(P,Q)$ are located at the points in the lower half of the P-plane where $\mathrm{i}\sqrt{n}P$ or $\mathrm{i}\frac{P}{\sqrt{n}}$ take non-negative integer values. At these points the factor $\kappa^{\mathrm{i}P\frac{(n+2)}{2\sqrt{n}}}$ in $Z^{(+)}(P,Q)$ 'resonates' with certain terms of the expansion (114) in the first term in (113). Therefore, the poles of the factor $Z^{(+)}(P,Q)$ in the second term can (and must) be cancelled by poles in appropriate terms of the expansion of the first term. For the 'perturbative' poles at $\mathrm{i}\sqrt{n}P = 0, 1, 2, \ldots$ this mechanism is evident in the semiclassical expression (69). The form similar to (113), together with this mechanism of the pole cancellation, was previously observed in the boundary sinh-Gordon model [35] and in the paperclip theory [1].

With the dual representation (101), (103) it is possible to make some quantitative predictions about the coefficients in expansions (114). In particular, using the approach outlined in [36] (see also [32, 37]), it is possible to obtain Coulomb gas integral representations for $f_{i,0}^{(-)}(P,Q)$ at the poles $P = \mathrm{i}\frac{m}{\sqrt{n}}, m = 0, 1, 2, \ldots$ of the first term in the sum (113). Unfortunately, for general m and i such integral formulae appear to be too cumbersome for any practical use. The sole exception is the case with $m = 0, i = 1$:

$$f_{1,0}^{(-)}\big|_{P=0} = -\frac{2^n\Gamma(1-n)}{4\pi^2 n}\,\frac{J(q)}{2\cos(\pi q)}, \tag{115}$$

where

$$\begin{aligned} J(q) = {} & \mathrm{e}^{\mathrm{i}\pi(q-\frac{n}{2})}J_1(q) + \mathrm{e}^{-\mathrm{i}\pi(q-\frac{n}{2})}J_1(-q) + \mathrm{e}^{\mathrm{i}\pi(q+\frac{n}{2})}J_2(q) \\ & + \mathrm{e}^{-\mathrm{i}\pi(q+\frac{n}{2})}J_2(-q) + \mathrm{e}^{\mathrm{i}\pi(q-\frac{n}{2})}J_3(q) + \mathrm{e}^{-\mathrm{i}\pi(q-\frac{n}{2})}J_3(-q) \end{aligned} \tag{116}$$

and

$$J_1 = \int_{0<w<v<u<2\pi} \left[2\sin\left(\frac{u-v}{2}\right)\right]^{-n-1} \left[4\sin\left(\frac{u-w}{2}\right)\sin\left(\frac{v-w}{2}\right)\right]^n \mathrm{e}^{\mathrm{i}q(v-u)}$$

$$J_2 = \int_{0<v<w<u<2\pi} \left[2\sin\left(\frac{u-v}{2}\right)\right]^{-n-1} \left[4\sin\left(\frac{u-w}{2}\right)\sin\left(\frac{w-v}{2}\right)\right]^n \mathrm{e}^{\mathrm{i}q(v-u)}$$

$$J_3 = \int_{0<v<u<w<2\pi} \left[2\sin\left(\frac{u-v}{2}\right)\right]^{-n-1} \left[4\sin\left(\frac{w-u}{2}\right)\sin\left(\frac{w-v}{2}\right)\right]^n \mathrm{e}^{\mathrm{i}q(v-u)}.$$

These integrals should be understood in a sense of analytical continuation in n from the domain of convergence ($-1 < \mathrm{Re}\, n < 0$). The phases in (116) are also worthy of note. They are generated by the additional boundary degree of freedom in (98) and make the sum of integrals (116) well defined. Were these phases absent $J(q)$ would depend on an auxiliary initial point of integration for the variables u, v, w. Here this point is chosen to be zero. For the given phase factors the sum of integrals (116) can be expressed in terms of the generalized hypergeometric function at unity to yield

$$f_{1,0}^{(-)}\big|_{P=0} = \frac{2^n\Gamma^2(-n)\Gamma\left(\frac{1}{2}+q\right)}{\Gamma\left(\frac{1}{2}+q-n\right)}\,{}_3F_2\left(\tfrac{1}{2}+q, -n, -n; 1, \tfrac{1}{2}+q-n; 1\right). \tag{117}$$

6. The IPH model for $n \to 0$

The dual description of the IPH model is especially useful in the strong coupling ($n \to 0$) regime. In particular for $n = 0$ the theory simplifies drastically and can be explored with full details.

6.1. Boundary amplitude

Here we consider the boundary amplitude (19) in the $n \to 0$ limit, assuming the parameters p, q (106) are fixed. For $n = 0$ the field X formally decouples in the dual Hamiltonian (98), (101). To be more precise the $n \to 0$ limit is, in fact, the classical limit for X. After the field redefinition,

$$\phi = \sqrt{n}X \tag{118}$$

this becomes particularly striking. Since quantum fluctuations of ϕ are suppressed as $n \to 0$ we may apply the saddle-point approximation to account their contribution. The theory is trivial in the bulk and, in consequence, the saddle point achieves at some constant classical field configuration which we shall denote as ϕ_0. Thus, in the limit $n \to 0$ we find

$$\hat{H}_{\mathrm{iph}}|_{n\to 0} = \hat{H}_{\mathrm{bulk}} - \frac{\mathrm{i}p}{\pi n R}\phi_0 + \mu_+\mathrm{e}^{-\phi_0} + h\sigma_3 + \mu_-\mathrm{e}^{\frac{\phi_0}{2}}\left[\sigma_+\,\mathrm{e}^{\mathrm{i}\frac{\sqrt{n+2}}{2}\tilde{Y}_B} + \sigma_-\,\mathrm{e}^{-\mathrm{i}\frac{\sqrt{n+2}}{2}\tilde{Y}_B}\right]_{n\to 0}, \tag{119}$$

with

$$h = -\frac{\mathrm{i}}{2}\left(\frac{q}{R} + \pi n\mu_+\,\mathrm{e}^{-\phi_0}\right). \tag{120}$$

One can observe that (119) is a Hamiltonian of the one-channel anisotropic Kondo model [38] and $n = 0$ corresponds to the so-called Toulouse point [39].[11] It is well known that the ground-state energy of the Kondo model suffers from a specific 'free-fermion' divergence at the Toulouse point. For this reason we prefer to keep n as an ultraviolet regulator in

[11] In the conventional Kondo model the field $\tilde{Y}$ emerges from the bosonization of electron degrees of freedom and is assumed to be compactified.

equation (119) rather than set it to zero. Note also that h (120) plays the role of an external magnetic field coupled with the impurity spin. The partition function of Kondo model is well known (see, e.g., [40]),

$$Z_{\text{Kondo}} = \frac{2\pi g_D\, \mathrm{e}^{-\frac{2\pi R}{n}U_{\text{eff}}(\phi_0)}}{\Gamma\big(\frac{1}{2}+2\mathrm{i}Rh+\pi R\mu_-^2\,\mathrm{e}^{\phi_0}\big)\Gamma\big(\frac{1}{2}-2\mathrm{i}Rh+\pi R\mu_-^2\,\mathrm{e}^{\phi_0}\big)}, \tag{121}$$

where

$$U_{\text{eff}}(\phi_0) = -\frac{\mathrm{i}p}{\pi R}\phi_0 + n\mu_+\,\mathrm{e}^{-\phi_0} + \frac{\mu_-^2}{n}\,\mathrm{e}^{\phi_0} + nE_0. \tag{122}$$

The first two terms in the effective potential U_{eff} came from the constant (ϕ_0-depended) terms in the Hamiltonian (119). The third term is the above-mentioned free-fermion divergence and E_0 is a non-universal constant with the dimension of energy. The saddle-point configuration ϕ_0 corresponds to the minima of the effective potential U_{eff}, which picks at

$$\sinh(\phi_*) = \frac{\mathrm{i}p}{\kappa}\,(\phi_0 \equiv -n\log r + \phi_*), \tag{123}$$

where we use the notations r and κ from equations (109), (110), (112) with $n \to 0$. Now we should take into account an effect of Gaussian fluctuations around the classical saddle-point configuration. We can write $\phi(\sigma,\tau) = \phi_0 + \delta\phi(\sigma,\tau)$. Then expanding the effective potential near the minima, we obtain the boundary potential in the Gaussian approximation,

$$U_{\text{eff}}(\phi_B) = \text{const} + \frac{1}{2\pi R}\sqrt{\kappa^2-p^2}(\delta\phi_B)^2 + O((\delta\phi_B)^3). \tag{124}$$

Finally, we can use the result from [22] to find the contribution of the Gaussian field $\delta\phi$ to the partition function. Combining all these ingredients together one arrives at the following result,

$$Z|_{n\to 0} = g_D^2 r^{-2\mathrm{i}p}\frac{2\sqrt{\pi}(\kappa^2-p^2)^{\frac{1}{4}}\Gamma(2\sqrt{\kappa^2-p^2})}{\Gamma\big(\frac{1}{2}-q+\mathrm{i}p\big)\Gamma\big(\frac{1}{2}+q+\sqrt{\kappa^2-p^2}\big)}(2\kappa\,\mathrm{e}^{\mathcal{E}})^{-\sqrt{\kappa^2-p^2}}\,\mathrm{e}^{-\frac{S(\kappa,p)}{n}}, \tag{125}$$

with

$$S = 2\sqrt{\kappa^2-p^2} + 2p\arcsin\left(\frac{p}{\kappa}\right), \tag{126}$$

and $\mathcal{E}$ is an arbitrary non-universal constant.

6.2. Boundary state

Not only the vacuum amplitude, but the whole boundary state (operator) can be also found in the limit $n \to 0$. Contrary to the previous discussion we shall keep now the zero-mode momentum P of order 1 for $n \to 0$. Hence, in particular, one should send $p = \sqrt{n}P/2 \to 0$ in expression (125) for the vacuum amplitude. With this limiting prescription X and Y sectors of the model are factorized completely:

$$\lim_{n\to 0}\mathbb{B} = \mathbb{B}^{(\text{x})}\otimes\mathbb{B}^{(\text{y})}. \tag{127}$$

Here $\mathbb{B}^{(\text{x})}$ is the boundary state operator corresponding to the Gaussian boundary interaction (124) with $p = 0$, and $\mathbb{B}^{(\text{y})}$ is the boundary state operator involving the Y-modes only. The operator $\mathbb{B}^{(\text{x})}$ admits a simple representation in terms of the oscillatory modes X_j (89) [41]:

$$\mathbb{B}^{(\text{x})} = g_D r^{\frac{\mathrm{i}P}{\sqrt{n}}}\sqrt{\frac{\kappa}{\pi}}\Gamma(2\kappa)\left(\frac{2\kappa}{\mathrm{e}}\right)^{-2\kappa}\mathrm{e}^{-\frac{P^2}{4\kappa}-\kappa\mathcal{E}_{\text{x}}}\exp\left(-\sum_{j=1}^{\infty}\frac{4}{j}\operatorname{arctanh}\left(\frac{j}{2\kappa}\right)X_{-j}X_j\right), \tag{128}$$

where $\mathcal{E}_x$ is some non-universal constant. The boundary state operator $\mathbb{B}^{(y)}$ does not have such a simple form as (128) for an arbitrary value of q. Nevertheless its whole spectrum is known. As was shown in [42], the eigenvectors of $\mathbb{B}^{(y)}$ in the Fock space of representation of the oscillatory modes Y_j can be labelled by two increasing sequences of positive integers

$$1 \leqslant n_1^+ < n_2^+ < \cdots < n_N^+, \qquad 1 \leqslant n_1^- < n_2^- < \cdots < n_N^- \qquad \text{with} \quad N \geqslant 0, \tag{129}$$

and the corresponding eigenvalues are given by[12]:

$$B^{(y)}_{(n^+|n^-)} = B_0^{(y)} \prod_{k=1}^{N} \frac{\left(q+\kappa-n_k^-+\frac{1}{2}\right)\left(q+\frac{\kappa}{2}+n_k^+-\frac{1}{2}\right)}{\left(q+\kappa+n_k^+-\frac{1}{2}\right)\left(q+\frac{\kappa}{2}-n_k^-+\frac{1}{2}\right)} \prod_{k=1}^{N} \frac{\left(q+n_k^+-\frac{1}{2}\right)\left(q+\frac{\kappa}{2}-n_k^-+\frac{1}{2}\right)}{\left(q-n_k^-+\frac{1}{2}\right)\left(q+\frac{\kappa}{2}+n_k^+-\frac{1}{2}\right)}, \tag{130}$$

where

$$B_0^{(y)} = g_D \frac{2\pi}{\Gamma\left(\frac{1}{2}-q\right)\Gamma\left(\frac{1}{2}+q+\kappa\right)} \left(\frac{\kappa}{\mathrm{e}}\right)^{\kappa} \mathrm{e}^{-\kappa\left(\frac{2}{n}+\mathcal{E}_y\right)},$$

and $\mathcal{E}_y$ again is an arbitrary constant.

A limiting behaviour of the boundary state operator (127) as $\kappa \to 0$ and $\kappa \to \infty$ deserves to be discussed in some details. As it follows from equation (130),

$$\lim_{\kappa\to 0} \mathbb{B}^{(y)} = (\mathrm{e}^{\mathrm{i}\pi q} + \mathrm{e}^{-\mathrm{i}\pi q}) g_D \mathbb{I}, \tag{131}$$

where $\mathbb{I}$ is the identity operator. Clearly, this boundary state operator corresponds to the Dirichlet boundary condition for the field Y such that its boundary values are constrained to two points,

$$Y_B|_{\mathrm{UV}} = \pm \frac{\pi}{\sqrt{2}}. \tag{132}$$

This conclusion is consistent with a naive interpolation of the boundary constraint (2) to $n = 0$. It is also clear that the field X is subjected by the free boundary condition in the short distance limit:

$$\lim_{\kappa\to 0} \mathbb{B}^{(x)} = g_D \delta(P) \exp\left(2\mathrm{i}\pi \sum_{j=1}^{\infty} \frac{X_{-j}X_j}{j}\right). \tag{133}$$

Now let us consider the infrared behaviour of the boundary state operator (127). As $\kappa \to \infty$,

$$\mathbb{B}^{(x)}|_{\kappa\to\infty} \to r^{\frac{\mathrm{i}P}{\sqrt{n}}} g_D \mathbb{I}\, \mathrm{e}^{-\kappa \mathcal{E}_x}. \tag{134}$$

This equation immediately suggests that the field X obeys the Dirichlet boundary condition in the infrared limit:

$$X_B|_{\mathrm{IR}} = \frac{1}{\sqrt{n}} \log r. \tag{135}$$

The infrared boundary condition for the field Y is far more interesting. As it follows from equation (130),

$$B^{(y)}_{(n^+|n^-)}\Big|_{\kappa\to\infty} \to g_D \kappa^{-q} \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-q\right)} \mathrm{e}^{-\kappa\left(\frac{2}{n}+\mathcal{E}_y\right)} \prod_{k=1}^{N} \frac{q+n_k^+-\frac{1}{2}}{q-n_k^-+\frac{1}{2}}. \tag{136}$$

[12] Equation (130) follows from formula (4.28) in [42], where $2p_{BLZ}$ and λ_{BLZ} should be replaced by the variables $q+\frac{\kappa}{2}$ and $-\mathrm{i}\sqrt{\frac{\kappa}{2\pi}}$ respectively.

If one looks at equations (134), (136) closely, several features stand out. First, it is expected *a priori* that the infrared behaviour of (127) is controlled by some scale-invariant boundary state (operator). More precisely, the scale dependence of the infrared boundary state (operator) is allowed in the form of a non-universal factor $\exp(-\text{const}\,\kappa)$ with $\kappa = E_* R$. Such factors are indeed presented in equations (134), (136). The additional scale-dependent factor κ^{-q} in (136) shows that when the RG 'time' $t = \log\kappa$ increases the field Y 'flows' uniformly with t. It can be made into a scale invariant fixed point by an appropriate redefinition of the RG transformation, namely by supplementing it with a formal field redefinition $(X, Y) \to \big(X, Y + \frac{\mathrm{i}}{\sqrt{2}}\delta t\big)$. This corresponds to the following modification of stress–energy tensor:

$$T_{\mathrm{IR}} = -(\partial X)^2 - (\partial Y)^2 + \frac{\mathrm{i}}{\sqrt{2}}\partial^2 Y, \qquad \bar{T}_{\mathrm{IR}} = -(\bar{\partial} X)^2 - (\bar{\partial} Y)^2 + \frac{\mathrm{i}}{\sqrt{2}}\bar{\partial}^2 Y, \tag{137}$$

or, in a stringy speak, to introducing a complex linear dilaton $D(\mathbf{X}) = \frac{\mathrm{i}}{\sqrt{2}}Y$. In view of the non-unitary of the IPH model in the Minkowski coordinates $(\sigma, \mathrm{i}\tau)$ the appearance of complex dilaton is not particularly surprising.

7. Exact boundary amplitude

In this section we propose an exact expression for the boundary amplitude (19). The expression is in terms of solutions of special second-order ordinary differential equation (ODE). Similar expressions are known in a number of integrable models of boundary interaction, beginning with the work of Dorey and Tateo [43] (see, e.g., [1] and references therein). Our proposal extends this relation to the IPH model. In this case no proof is yet available, but we will show in this section that the proposed expression reproduces all the properties of the amplitude described above.

7.1. Differential equation

Consider the ordinary second-order differential equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - p^2 + 2q\kappa\,\mathrm{e}^x + \kappa^2(\mathrm{e}^{2x} + \mathrm{e}^{-nx})\right]\Psi(x) = 0, \tag{138}$$

where p, q are related to the components of $\mathbf{P} = (P, Q)$ in (19) as in equation (106), and $\kappa = E_* R$. With this identification in mind, below we always assume that κ is real and positive.

Let $\Psi_-(x)$ be the solution of (138) which decays when x goes to $-\infty$ along the real axis, and $\Psi_+(x)$ be another solution of (138), the one which decays at large positive x. We fix normalizations of these two solutions as follows,

$$\Psi_- \to (2\kappa)^{-\frac{1}{2}} \exp\left(\frac{nx}{4} - \frac{2\kappa}{n}\,\mathrm{e}^{-\frac{nx}{2}}\right) \qquad \text{as} \quad x \to -\infty, \tag{139}$$

and

$$\Psi_+ \to (2\kappa)^{-q-\frac{1}{2}} \exp\left\{-\big(q + \tfrac{1}{2}\big)x - \kappa\,\mathrm{e}^x\right\} \qquad \text{as} \quad x \to +\infty. \tag{140}$$

Let

$$\mathrm{W}[\Psi_+, \Psi_-] \equiv \Psi_+ \frac{\mathrm{d}}{\mathrm{d}x}\Psi_- - \Psi_- \frac{\mathrm{d}}{\mathrm{d}x}\Psi_+ \tag{141}$$

be the Wronskian of these two solutions. Then, our proposal for the function (19) is

$$Z = g_D^2 r^{-2\mathrm{i}p} \frac{\sqrt{2\pi}}{\Gamma\big(\frac{1}{2} - q + \mathrm{i}p\big)} \mathrm{W}[\Psi_+, \Psi_-]. \tag{142}$$

To make more clear the motivations behind this proposal let us discuss some properties of the solutions of the differential equation (138).[13]

7.2. *Small κ expansion*

The ODE (138) has the form of one-dimensional Schrödinger equation with the potential defined by the last three terms in the bracket in (138). When κ goes to zero this potential develops a wide plateau at

$$\frac{2}{n}\log\kappa \ll x \ll -\log\kappa, \tag{143}$$

where its value is close to $-p^2$. In this domain each of the solutions Ψ_+ and Ψ_- is a combination of two plane waves:

$$\Psi_\pm = D_\pm(p,q|\kappa)\,\mathrm{e}^{\mathrm{i}px} + D_\pm(-p,q|\kappa)\,\mathrm{e}^{-\mathrm{i}px}. \tag{144}$$

Let us consider the amplitude D_+ in (144) for the solution Ψ_+ first. In this case it is convenient to change the variable, $x = y - \log(2\kappa)$, and bring the equation (138) to the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}y^2} - p^2 + q\,\mathrm{e}^y + \frac{1}{4}\,\mathrm{e}^{2y} + \delta V_+\right]\Psi(x) = 0, \tag{145}$$

with

$$\delta V_+ = 2^n\kappa^{n+2}\,\mathrm{e}^{-ny}. \tag{146}$$

Let us assume for the moment that

$$-2 < \mathrm{Re}\,n < 0. \tag{147}$$

Then the term δV_+ can be considered as a small perturbation for any $-\infty < y < +\infty$. At the zero perturbative order we dismiss δV_+ in (145). It gives an equation which can be brought to the form of Kummer's equation [24] by a simple change of variables. The unperturbed solution reads explicitly,

$$\Psi_+^{(0)}(y) = \mathrm{e}^{-\frac{\mathrm{e}^y}{2}}\,\mathrm{e}^{-\mathrm{i}py}\,U\left(\tfrac{1}{2} + q - \mathrm{i}p, 1 - 2\mathrm{i}p, \mathrm{e}^y\right). \tag{148}$$

The normalization of $\Psi_+^{(0)}$ is chosen to match the asymptotic condition (140). Now one can systematically develop the standard perturbation theory for the solution Ψ_+. Therefore $D_+(p,q|\kappa)$ (144) admits the following power series expansion:

$$D_+(p,q|\kappa) = (2\kappa)^{\mathrm{i}p}\frac{\Gamma(-2\mathrm{i}p)}{\Gamma\left(\frac{1}{2} + q - \mathrm{i}p\right)}\left(1 + \sum_{i=1}^{\infty} d_i^{(+)}(p,q)\kappa^{i(n+2)}\right). \tag{149}$$

The first-order perturbative coefficient $d_1^{(+)}$ can be calculated in the closed form. Indeed, it is represented by the following integral:

$$d_1^{(+)}(p,q) = \frac{2^n\Gamma\left(\frac{1}{2} + q - \mathrm{i}p\right)}{\Gamma(1 - 2\mathrm{i}p)}\int_0^\infty \mathrm{d}z\, z^{-n-1}\,\mathrm{e}^{-z} \\ \times M\left(\tfrac{1}{2} + q - \mathrm{i}p, 1 - 2\mathrm{i}p, z\right)U\left(\tfrac{1}{2} + q + \mathrm{i}p, 1 + 2\mathrm{i}p, z\right), \tag{150}$$

[13] Equation (138) differs in the term $2q\kappa\,\mathrm{e}^x$ only from the ODE describing the boundary sinh-Gordon model (see equation (40) from the work [46] with $\beta^2 < 0$). Because of this the subsequent analysis is parallel to that from the unpublished work of Al B Zamolodchikov [35].

which converges for $-1 < \mathrm{Re}\, n < 0$ and can be expressed in terms of the generalized hypergeometric function at unity (see equation (7.625) in [49]):

$$d_1^{(+)}(p,q) = \frac{2^n \Gamma\big(\frac{1}{2}+q-\mathrm{i}p\big)\Gamma(-2\mathrm{i}p-n)\Gamma(-n)}{\Gamma(1-2\mathrm{i}p)\Gamma\big(\frac{1}{2}+q-\mathrm{i}p-n\big)} \times {}_3F_2\big(\tfrac{1}{2}+q-\mathrm{i}p, -n, -n-2\mathrm{i}p; 1-2\mathrm{i}p, \tfrac{1}{2}-n+q-\mathrm{i}p : 1\big). \qquad (151)$$

Evidently the power series (149) converges for any complex κ and defines an entire function of κ^{n+2} for complex $n \neq -2+\frac{1}{k}(k = 1,2,3,\ldots)$ from the strip (147).[14] There is no reason to expect the convergence for real positive n. However it is expected that for $n > 0$ equation (149) is valid in a sense of $\kappa \to 0$ asymptotic expansion with the zero convergence radius.

Similarly one can derive an expansion of the amplitude D_- (144) for the solution Ψ_-. In this case we make a change of variable $x = \frac{2}{n}z + \frac{2}{n}\log\big(\frac{2\kappa}{n}\big)$; equation (138) then takes the form,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \left(\frac{2p}{n}\right)^2 + \mathrm{e}^{-2z} + \delta V_-\right]\Psi(x) = 0, \qquad (152)$$

with

$$\delta V_- = \frac{4q}{n}\left(\frac{2\kappa}{n}\right)^{\frac{n+2}{n}} \mathrm{e}^{\frac{2z}{n}} + \left(\frac{2\kappa}{n}\right)^{\frac{2(n+2)}{n}} \mathrm{e}^{\frac{4z}{n}}. \qquad (153)$$

For

$$\mathrm{Re}\, n < -2, \qquad (154)$$

δV_- can be treated as a perturbation for any real z. The small parameter now is $\kappa^{\frac{n+2}{n}}$. At the zero order one has

$$\Psi_-^{(0)}(z) = \sqrt{\frac{2}{\pi n}}\, K_{\frac{2\mathrm{i}p}{n}}(\mathrm{e}^{-z}), \qquad (155)$$

where $K_\nu(x)$ is the conventional MacDonald function. Again, the normalization of the solution is chosen to match (139). The perturbative expansion for $D_-(p,q|\kappa)$ (144) has a form

$$D_-(p,q|\kappa) = \left(\frac{\kappa}{n}\right)^{-\frac{2\mathrm{i}p}{n}} \frac{\Gamma\big(\frac{2\mathrm{i}p}{n}\big)}{\sqrt{2\pi n}} \left(1 + \sum_{j=1}^{\infty} d_j^{(-)}(p,q)\kappa^{j\frac{(n+2)}{n}}\right). \qquad (156)$$

This series converges for any complex κ if n belongs to the complex half-plane (154). For real positive n it is an asymptotic series with the zero convergence radius. The low order coefficient $d_1^{(-)}$ can be easily obtained by direct perturbative calculations:

$$d_1^{(-)}(p,q) = 2q\left(\frac{2}{n}\right)^{\frac{2(n+1)}{n}} \frac{\Gamma\big(\frac{1}{2}+\frac{1}{n}\big)\Gamma\big(-\frac{1}{n}\big)}{4\sqrt{\pi}} \frac{\Gamma\big(-\frac{1}{n}+\frac{2\mathrm{i}p}{n}\big)}{\Gamma\big(1+\frac{1}{n}+\frac{2\mathrm{i}p}{n}\big)}. \qquad (157)$$

The asymptotic expansions (149), (156) are useful in studying small κ behaviour of the Wronskian (141). Indeed for small κ the Wronskian can be calculated at any point from the domain (143), where $\Psi_\pm$ are combinations of two plane waves (144). Hence the Wronskian (141) is written in terms of the coefficients in $D_\pm(p,q)$ as follows,

$$\mathrm{W}[\Psi_+,\Psi_-] = -2\mathrm{i}p(D_+(p,q|\kappa)D_-(-p,q|\kappa) - D_+(-p,q|\kappa)D_-(p,q|\kappa)). \qquad (158)$$

[14] For $q = 0$ the function $D_+(p,0|\kappa)$ coincides up to the overall factor with a vacuum eigenvalue of the Q-operator from the work [42] upon the parameter identifications: $\beta_{BLZ}^2 = -\frac{n}{2}$, $p_{\mathrm{BLZ}} = -\frac{\mathrm{i}}{2}p$ and $\lambda_{\mathrm{BLZ}}^2 = -\frac{2^{-n-2}}{\Gamma^2(1+\frac{n}{2})}\kappa^{n+2}$. In this connection, it is pertinent to note that the higher-order coefficients in (149) can be calculated by the method developed in the appendix of the work [50].

Then (142) leads to the short distance expansion of the partition function of the form (113), where P, Q are related to p, q through equations (106). The coefficient $f_{i,j}^{(\pm)}$ in the double power series (114) is simply expressed in terms of $d_k^{(\pm)}$:

$$f_{i,j}^{(\pm)} = d_i^{(+)}(\pm p, q) d_j^{(-)}(\mp p, q), \tag{159}$$

with $d_0^{(\pm)} = 1$. In particular $f_{1,0}^{(\pm)}(P, Q)|_{P=0} = d_1^{(+)}(0, q)$, where $d_1^{(+)}(p, q)$ is given by (151). This is in a complete agreement with the result (117) of perturbative calculation in the IPH model.

7.3. Semiclassical domain $\kappa \ll 1 \ll n$

When κ is small and $n \gg 1$ the above expansions have to be collected. This regime corresponds to the semiclassical domain in the IPH model considered in section 3.

First, let us consider the case when p and q in (138) are order of 1; it corresponds to the case of 'light' insertion in (54) with $(P, Q) \sim \frac{1}{\sqrt{n}}$ (61). In this regime the term $\kappa^2\,\mathrm{e}^{-nx}$ in equation (138) has the effect of a rigid wall at some point x_0, i.e., to the right from this point, for $x - x_0 \gg \frac{1}{n}$, this term is negligible, but to the left from x_0 it grows very fast, so that for $x < x_0$ the solution Ψ_- is essentially zero. More precisely, when x is above but close to x_0 the solution Ψ_- is approximated by a linear function,

$$\Psi_-(x) \approx \alpha_0(x - x_0), \tag{160}$$

where the position of the wall x_0 and the slope α_0 are given by

$$x_0 = \frac{2}{n}\log\left(\frac{\kappa\,\mathrm{e}^{\gamma_E}}{n}\right) + o\left(\tfrac{1}{n}\right), \qquad \alpha_0 = \sqrt{\frac{n}{2\pi}}(1 + o(1)), \tag{161}$$

as $n \to \infty$. Here γ_E is Euler's constant. We will explain these relations shortly. Wronskian (141) is easily determined. Taking, for instance, any point x close to the right of the wall where both equations (148) and (160) are valid, one finds it equal α_0 times expression (148) evaluated at $\mathrm{e}^{y_0} = 2\kappa\mathrm{e}^{x_0}$, or, due to equation (161), at $\mathrm{e}^{y_0} = 2n\lambda(1 + o(1))$, where $\lambda = \left(\frac{\kappa}{n}\right)^{\frac{n+2}{n}}$. This leads exactly to (67).

The case of 'heavy' insertion, $(P, Q) \sim \sqrt{n}$ (72), can be handled similarly. The term $\kappa^2\mathrm{e}^{-nx}$ still can be treated as a wall at x_0, in the sense that at $x - x_0 \gg \frac{1}{n}$ its effect is negligible, but it dominates at $x_0 - x \gg \frac{1}{n}$. To study the vicinity of the wall we make a change of the variable, $x = \frac{2}{n}z + \frac{2}{n}\log\left(\frac{2\kappa}{n}\right)$, and bring equation (138) to form (152). With (152) it is evident that the wall is located at $x_0 = \frac{2}{n}\log\left(\frac{2\kappa}{n}\right) + \frac{2}{n}z_0 + o\left(\frac{1}{n}\right)$, where z_0 is some n-independent constant. Close to the walls position, i.e., at

$$|x - x_0| \ll 1 \tag{162}$$

δV_- (153) can be approximated by the constant

$$\delta V_- \approx \frac{8q}{n}\lambda + 4\lambda^2, \qquad \text{where} \quad \lambda = \left(\frac{\kappa}{n}\right)^{\frac{n+2}{n}}. \tag{163}$$

Therefore in domain (162) the solution Ψ_- is approximated by the MacDonald function,

$$\Psi_- \approx \sqrt{\frac{2}{\pi n}}\,K_{2\mathrm{i}\nu_0}(\mathrm{e}^{-z}), \tag{164}$$

with $\nu_0 = \frac{1}{n}\sqrt{p^2 - 2qn\lambda - (n\lambda)^2}$. The parameter ν_0 here is essentially the same as in equation (74). The normalization factor in front of K is fixed by matching (164) to the asymptotic condition (139). For $\nu_0 \to 0$ and $z \gg 1$ (164) reduces to (160). The Wronskian (141) can be evaluated in the domain $\frac{1}{n} \ll x - x_0 \ll 1$, where both equations (148) and (164) are valid; the result is exactly (69) with the extra factor (78) added.

7.4. Large κ expansion

The case of large κ is expected to describe the infrared limit of the IPH model. For the differential equation (138) it is the domain of validity of the WKB approximation. Applying the standard WKB iteration scheme [51] one finds for the Wronskian (141),

$$\log W = \lim_{L\to+\infty}\left\{\kappa\int_{-\infty}^{L}\mathrm{d}x\big(\mathcal{P}(x)-\mathrm{e}^{-\frac{nx}{2}}\big)-\kappa\,\mathrm{e}^{L}-qL\right\}+\frac{1}{8\kappa}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(\mathcal{P}'(x))^2}{\mathcal{P}^3(x)}+\cdots, \tag{165}$$

where

$$\mathcal{P}(x)=\sqrt{\mathrm{e}^{2x}+\mathrm{e}^{-nx}+\frac{2q}{\kappa}\,\mathrm{e}^{x}-\frac{p^2}{\kappa^2}}. \tag{166}$$

The subtraction term $\mathrm{e}^{-\frac{nx}{2}}$ in the integrand derives from the asymptotic conditions (139) and ensures convergence of the first integral in (165) as $x\to-\infty$. The terms depending on L cancel the divergence of this integral as $x\to+\infty$. It follows from the asymptotic condition (140). Equation (165) generates asymptotic expansion of the partition function (19),

$$\log Z\simeq\log Z_{\mathrm{IR}}-\sum_{s=1}^{\infty}\frac{I_s^{(\mathrm{norm})}(P,Q)}{2\sin\big(\frac{\pi s}{n+2}\big)}\left(\frac{n}{2\kappa}\right)^{s}, \tag{167}$$

where

$$Z_{\mathrm{IR}}=g_D^2\,r^{-2\mathrm{i}p}\kappa^{-q}2^{\frac{nq}{n+2}}\frac{\sqrt{2\pi}}{\Gamma\big(\frac12-q+\mathrm{i}p\big)}\exp\left(-\frac{\kappa}{\kappa_0}\right), \tag{168}$$

and the constant κ_0 reads explicitly,

$$\kappa_0=-\frac{2\sqrt{\pi}}{\Gamma\big(-\frac{n}{2(n+2)}\big)\Gamma\big(1-\frac{1}{n+2}\big)}. \tag{169}$$

The functions $I_s^{(\mathrm{norm})}$ in (167) are polynomials of the variables P^2 and Q of the degree $s+1$. Their highest-order terms follow from the first integral in (165),

$$I_s^{(\mathrm{norm})}=\frac{(-1)^s\sqrt{\pi}}{2s}\sum_{2l+m=s+1}\frac{2^m(n+2)^{\frac{m}{2}}\Gamma\big(\frac12-l+\frac{(n+1)s}{n+2}\big)}{n^{s-l}\Gamma\big(-\frac{s}{n+2}\big)}\frac{P^{2l}Q^m}{m!l!}+\cdots, \tag{170}$$

which is in agreement (up to an overall normalization) with the highest-order terms of the vacuum eigenvalues (95). It is also straightforward to generate the full polynomials evaluating integral (165) order by order in κ^{-1}. With the adjusted overall normalization of the local IM, this calculation exactly reproduces the vacuum eigenvalues (95).

7.5. $n\to0$ limit

There is no problem with the limit $n\to0$ for the differential equation (138) and for the solution Ψ_+ (140). If $n=0$ the equation turns out to be of Kummer's type [24] and

$$\Psi_+(x)|_{n=0}=\mathrm{e}^{-\kappa\mathrm{e}^{x}}(2\kappa\,\mathrm{e}^{x})^{-\sqrt{\kappa^2-p^2}}U\big(\tfrac12+q-\sqrt{\kappa^2-p^2},1-2\sqrt{\kappa^2-p^2},2\kappa\,\mathrm{e}^{x}\big). \tag{171}$$

However, the asymptotic condition (139) is singular as $n\to0$ and the limiting behaviour of the solution Ψ_- is a more delicate issue. To proceed with the limit we will use the WKB approximation for Ψ_-:

$$\Psi_-^{(\mathrm{wkb})}(x)=\frac{1}{\sqrt{2\kappa\mathcal{P}(x)}}\exp\left\{\kappa\int_{-\infty}^{x}\mathrm{d}t\big(\mathcal{P}(t)-\mathrm{e}^{-\frac{nt}{2}}\big)-\tfrac{2\kappa}{n}\,\mathrm{e}^{-\frac{nx}{2}}\right\}, \tag{172}$$

where $\mathcal{P}(x)$ is given by (166). Let us consider the argument $\{\cdots\}$ of the exponential in (172) as $n \to 0$. It is easy to see that $\{\cdots\} \to F(x) - \frac{1}{n}S(\kappa, p)$, where x-independent function $S(\kappa, p)$ has the form (126), while $F(x) \to x\sqrt{\kappa^2 - p^2}$ as $x \to +\infty$. Thus we see that

$$\Psi_-(x) \to \Psi_-^{(\mathrm{reg})}(x) 2^{-\frac{1}{2}}(\kappa^2 - p^2)^{-\frac{1}{4}} \,\mathrm{e}^{-\frac{S(\kappa,p)}{n}} \qquad \text{as} \quad n \to 0, \tag{173}$$

where $\Psi_-^{(\mathrm{reg})}$ is a solution of the differential equation (138) with $n = 0$ such that $\Psi_-^{(\mathrm{reg})} \to \mathrm{e}^{x\sqrt{\kappa^2-p^2}}$ as $x \to -\infty$. It is straightforward now to calculate the boundary amplitude (142). The result coincides with equation (125).

8. Integrable structures of the theory

We have now seen that the expression (142) passed successfully all available checks. In this section we will discuss properties of the vacuum amplitude (142) to reveal some standard integrable structures [11, 13, 14] of the theory.

8.1. Thermodynamic Bethe Ansatz equations

It is well known from the global theory of linear ordinary differential equations [44, 45] that monodromic coefficients of linear ODE, like the Wronskian (141), satisfy the difference equations as functions of parameters. For equation (138) with $q = 0$ and $n < 0$ the corresponding difference equations were derived in the work [46] (see also [47, 48]). The case $q = 0$, $n > 0$ was studied in the unpublished paper [35]. Here we describe basic properties of the Wronskian (141) for $n > 0$ and an arbitrary q. For this purpose, it is convenient to modify slightly our notations. In particular we introduce now the parameter θ such that

$$\mathrm{e}^{\theta} = \frac{\kappa}{\kappa_0}, \tag{174}$$

where the constant κ_0 is given by (169). Also, to emphasize the dependence on θ and $q = \frac{\sqrt{n+2}}{2}Q$ explicitly, we will denote the Wronskian (141) by $\mathrm{W}_q(\theta)$. Note that there is no need to indicate the dependence on $p = \frac{\sqrt{n}}{2}P$ explicitly.

The following properties of the function $\mathrm{W}_q(\theta)$ in the case $n > 0$ are readily made using the general theory [44] and our previous analysis:

- $\mathrm{W}_q(\theta)$ is entire function of θ. It is also an entire function of the parameters q and p.
- $\mathrm{W}_q(\theta)$ satisfies the so-called quantum Wronskian condition (see appendix B for details):

$$\mathrm{W}_{-q}\left(\theta - \frac{\mathrm{i}\pi}{2}\right)\mathrm{W}_q\left(\theta + \frac{\mathrm{i}\pi}{2}\right) - \mathrm{W}_q\left(\theta + \frac{\mathrm{i}\pi a}{2}\right)\mathrm{W}_{-q}\left(\theta - \frac{\mathrm{i}\pi a}{2}\right) = \mathrm{e}^{-\mathrm{i}\pi q}, \tag{175}$$

  where $a = \frac{n-2}{n+2}$.
- As a function of the complex variable θ, $\mathrm{W}_q(\theta)$ does not have zeros in the strip $|\mathrm{Im}\,\theta| < \frac{\pi}{2} + \epsilon$ for some finite $\epsilon > 0$.
- For $|\mathrm{Im}\,\theta| < \frac{\pi}{2} + \epsilon$ and $\mathrm{Re}\,\theta \to +\infty$:

$$\mathrm{W}_q(\theta) = \left(2^{\frac{n}{n+2}}\kappa_0\,\mathrm{e}^{\theta}\right)^{-q} \exp(-\mathrm{e}^{\theta} + O(\mathrm{e}^{-\theta})). \tag{176}$$

- For $\mathrm{Re}\,\theta \to -\infty$:

$$\mathrm{W}_q(\theta) \to L_q(-\mathrm{i}p)\,\mathrm{e}^{\frac{\mathrm{i}p\theta(n+2)}{n}} + L_q(\mathrm{i}p)\,\mathrm{e}^{-\frac{\mathrm{i}p\theta(n+2)}{n}}, \tag{177}$$

  where

$$L_q(h) = \sqrt{\frac{n}{2\pi}}\,2^{-h} n^{\frac{2h}{n}} (\kappa_0)^{-\frac{(n+2)h}{n}} \frac{\Gamma(2h)\Gamma\left(1 + \frac{2h}{n}\right)}{\Gamma\left(\frac{1}{2} + q + h\right)}.$$

Introduce the functions $\varepsilon_\pm(\theta)$:

$$\mathrm{e}^{\pm \mathrm{i}\pi q}\mathrm{W}_q\left(\theta \pm \frac{\mathrm{i}\pi}{2}\right)\mathrm{W}_{-q}\left(\theta \mp \frac{\mathrm{i}\pi}{2}\right) = 1 + \mathrm{e}^{-\varepsilon_\pm(\theta)}. \tag{178}$$

With the foregoing analytical conditions it is straightforward (see [52, 53]) to transform the difference equation (175) into a system of two integral equations for $\varepsilon_\pm(\theta)$:

$$2\sin\left(\frac{2\pi}{n+2}\right)\mathrm{e}^{\alpha} = \varepsilon_\pm(\alpha) \pm \frac{4\pi\mathrm{i}q}{n+2} + \int_{-\infty}^{+\infty}\frac{\mathrm{d}\beta}{2\pi}\{\varphi_{++}(\alpha-\beta) \times \log(1+\mathrm{e}^{-\varepsilon_\pm(\beta)}) + \varphi_{+-}(\alpha-\beta)\log(1+\mathrm{e}^{-\varepsilon_\mp(\beta)})\}, \tag{179}$$

where the kernels are given by $\varphi_{\sigma\sigma'}(\alpha) = -\mathrm{i}\partial_\alpha \log S_{\sigma\sigma'}(\alpha)$ with

$$S_{++}(\alpha) = \frac{\sinh\left(\frac{\alpha}{2} - \frac{\mathrm{i}\pi}{n+2}\right)}{\sinh\left(\frac{\alpha}{2} + \frac{\mathrm{i}\pi}{n+2}\right)}, \qquad S_{+-}(\alpha) = S_{++}(\mathrm{i}\pi - \alpha). \tag{180}$$

The integral equations (179) should be supplemented by an asymptotic condition for $\varepsilon_\pm(\alpha)$ as $\alpha \to -\infty$ which follows from equation (177). For instance, for $\mathrm{Im}\,p < 0$:

$$\varepsilon_\pm(\alpha)|_{\alpha\to-\infty} \to \mathrm{i}p\frac{2(n+2)}{n}\alpha \mp \mathrm{i}\pi q + \log(L_q(\mathrm{i}p)L_{-q}(\mathrm{i}p)). \tag{181}$$

As it follows from equation (178) the Wronskians $\mathrm{W}_{\pm q}(\theta)$ are expressed in terms of the solutions of (179), (181) by means of relations:

$$\log \mathrm{W}_{\pm q}(\theta) = \mp q \log\left(2^{\frac{n}{n+2}}\kappa_0\,\mathrm{e}^{\theta}\right) - \mathrm{e}^{\theta} + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\pi} \times\left\{\frac{\log(1+\mathrm{e}^{-\varepsilon_+(\alpha)})\log(1+\mathrm{e}^{-\varepsilon_-(\alpha)})}{\cosh(\theta-\alpha)} \mp \mathrm{i}\log\left(\frac{1+\mathrm{e}^{-\varepsilon_+(\alpha)}}{1+\mathrm{e}^{-\varepsilon_-(\alpha)}}\right)\frac{\mathrm{e}^{\alpha-\theta}}{\cosh(\theta-\alpha)}\right\}. \tag{182}$$

It is remarkable that the system (179) differs only in the structure of 'source terms' from the TBA system associated with the complex sinh-Gordon model, which is a non-compact version of the Lund–Regge model [7, 54]. It is well to bear in mind that the classical Lund–Regge model, introduced in [55], is a representative of the AKNS soliton hierarchy. Note also that if $q = 0$, then $\epsilon_+ = \epsilon_-$ and (179) turns into the integral equation which describes a vacuum boundary amplitude in the boundary sinh-Gordon model [35].

8.2. $\mathbb{T}$-*operator*

Let us consider the differential equation in the form (145) and the solution Ψ_- (139) as a function of the variable $y = x + \log(2\kappa)$ and the parameters θ (174) and q, i.e., $\Psi_- = \Psi_-(\theta, q; y)$. Note that the transformations $\theta \to \theta \pm \frac{2\pi\mathrm{i}}{n+2}$ leave ODE (145) unchanged. Hence, the functions $\Psi_-\big(\theta \pm \frac{2\pi\mathrm{i}}{n+2}, q; y\big)$ solve this equation as well as $\Psi_-(\theta, q; y)$, and the Wronskian

$$T(\theta, q) = \mathrm{i}\mathrm{W}\left[\Psi_-\left(\theta + \frac{2\pi\mathrm{i}}{n+2}, q; y\right), \Psi_-\left(\theta - \frac{2\pi\mathrm{i}}{n+2}, q; y\right)\right] \tag{183}$$

does not depend on the variable y. It can be shown (see appendix B for details) that function $T(\theta, q)$ is expressed through the Wronskian $\mathrm{W}_q(\theta)$ (141) as

$$\mathrm{W}_q(\theta)T(\theta, q) = \mathrm{W}_q\left(\theta + \frac{2\pi\mathrm{i}}{n+2}\right) + \mathrm{W}_q\left(\theta - \frac{2\pi\mathrm{i}}{n+2}\right), \tag{184}$$

and

$$T\left(\theta + \mathrm{i}\pi\frac{n}{n+2}, q\right) = T(\theta, -q). \tag{185}$$

Since the vacuum amplitude (142) differs by θ-independent factor from the Wronskian[15], one can replace W_q by Z in (184). It is also clear that the function $T(\theta, q)$ admits $\theta \to +\infty$ asymptotic expansion similar to (167). The foregoing in turn leads us to

Conjecture. *There exists an operator $\mathbb{T}$ which acts invariantly in the Fock space $\mathcal{F}_{\mathbf{P}}$ and satisfies the following conditions:*

- *Being considered as a function of the parameter $\lambda \sim e^{\frac{\theta(n+2)}{n}}$ (16), $\mathbb{T} = \mathbb{T}(\lambda)$ admits the convergent power series expansion of the form*

$$\mathbb{T}(\lambda) = 2\cosh\left(\frac{\pi P}{\sqrt{n}}\right) + \sum_{k=1}^{\infty} \mathbb{G}_k \lambda^k. \tag{186}$$

The coefficients $\mathbb{G}_k$ in (186) satisfy the condition,

$$\mathbb{R}\,\mathbb{G}_k\,\mathbb{R} = (-1)^k \mathbb{G}_k, \tag{187}$$

where the operator $\mathbb{R} : \mathcal{F}_{(P,Q)} \to \mathcal{F}_{(P,-Q)}$ flips the overall sign of the field ∂Y, i.e., $\mathbb{R}\,\partial Y\,\mathbb{R} = -\partial Y$.

- *The operator $\mathbb{T}(\lambda)$ commutes with all the IM from AKNS series and, hence,*

$$\big[\mathbb{T}(\lambda'), \mathbb{B}(\lambda)\big] = 0. \tag{188}$$

Here the boundary state operator $\mathbb{B}$ is understood as a multi-valued function of the spectral parameter (16). The eigenvalue of $\mathbb{T}$ corresponding to the Fock vacuum $|\mathbf{P}\rangle \in \mathcal{F}_{\mathbf{P}}$ coincides with $T(\theta, q)$ (183).

- *The boundary state operators $\mathbb{B}$ and $\mathbb{T}$ satisfy the $T - Q$ Baxter equation (18).*
- *The operator $\mathbb{T}(\lambda)$ is a generating function of the AKNS series of local IM. In more exact terms the local IM are generated in the large λ asymptotic series expansion of $\mathbb{T}(\lambda)$:*

$$\mathbb{T}(\lambda) = \exp\big(-2\pi\nu\big(\lambda\,\mathrm{e}^{\frac{\mathrm{i}\pi}{n}}\big)\big) + \exp\big(2\pi\nu\big(\lambda\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{n}}\big)\big), \tag{189}$$

where

$$i\nu(\lambda) \simeq -\frac{\Gamma\left(\frac{n+4}{2n+4}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+3}{n+2}\right)}\lambda^{\frac{n}{n+2}} - \frac{Q}{2\sqrt{n+2}} + \frac{1}{2\pi}\sum_{s=1}^{\infty}\left(\frac{R}{2\lambda^{\frac{n}{n+2}}}\right)^s \mathbb{I}_s^{(\mathrm{norm})} \qquad \text{as} \quad \lambda \to +\infty. \tag{190}$$

The local IM $\mathbb{I}_s^{(\mathrm{norm})}$ in (190) are normalized in accordance with the condition,

$$\mathbb{I}_s^{(\mathrm{norm})} = \frac{i^{s+1}}{n^{\frac{s+1}{2}}\sqrt{\pi}} \int_0^{2\pi R} \mathrm{d}\tau \left\{ \sum_{2l+m=s+1} \frac{\Gamma\left(\frac{1}{2} - l + \frac{(n+1)s}{n+2}\right)}{\Gamma\left(1 - \frac{s}{n+2}\right)} \times 2^{m+s-1}\left(\frac{n+2}{n}\right)^{\frac{m}{2}-1} \frac{(\partial X)^{2l}}{l!}\frac{(\partial Y)^m}{m!} + \cdots \right\}, \tag{191}$$

where omitted terms contain higher derivatives of $\partial\mathbf{X}$.[16]

[15] Here we assume that the parameter r in (142) does not depend on θ. This assumption follows from the normalization condition (65).

[16] Note that the local IM $\mathbb{I}_s^{(\mathrm{norm})}$ are defined unambiguously with the normalization condition (191).

Eventually, the operator $\mathbb{T}(\lambda)$ should be viewed as a quantum version of the transfer matrix $T(\lambda)$ (50) for the AKNS linear problem (46) with $j = \frac{1}{2}$. Following the line of [11] it seems possible to express the coefficients $\mathbb{G}_k$ in the power series expansion (186) in terms of $2k$-fold integrals over chiral vertex operators involving the holomorphic component of field $\mathbf{X}$. Since the nonlocal operators $\mathbb{G}_k$ commute among themselves $[\mathbb{G}_k, \mathbb{G}_m] = 0$ and also commute with all the local IM $\mathbb{I}_s$, $[\mathbb{G}_k, \mathbb{I}_s] = 0$, they are called the nonlocal integrals of motion. Unfortunately, the 'explicit' formulae for nonlocal IM are not particularly useful either to prove the above-listed properties of the operator $\mathbb{T}$ or for calculations of its spectrum. For this reason, we do not present these formulae here.

Finally we note that the vacuum eigenvalues $G_k^{(\mathrm{vac})}$ of nonlocal IM,

$$\mathbb{G}_k|\mathbf{P}\rangle = G_k^{(\mathrm{vac})}|\mathbf{P}\rangle, \tag{192}$$

can be algebraically expressed in terms of the coefficients $d_j^{(-)}(p, q)$ in the formal power series expansion (156). For instance,

$$\begin{aligned} G_1^{(\mathrm{vac})} &= -4n^{\frac{n+2}{n}} \sin\left(\tfrac{\pi}{n}\right) \sin\left(\tfrac{\pi(2\mathrm{i}p+1)}{n}\right) d_1^{(-)}(p, q) \\ &= -\frac{2^{\frac{2}{n}}\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}{\sqrt{\pi}\,\Gamma\left(1+\frac{1}{n}\right)} \frac{8\pi^2 \frac{q}{n}}{\Gamma\left(1+\frac{1}{n}-\frac{2\mathrm{i}p}{n}\right)\Gamma\left(1+\frac{1}{n}+\frac{2\mathrm{i}p}{n}\right)}. \end{aligned} \tag{193}$$

8.3. Commuting families in the quantum AKNS hierarchy

In [11, 56] the quantization procedure for the KdV hierarchy was developed. In the BLZ approach quantum transfer matrices are defined in terms of certain monodromy matrices associated with $2j + 1$ dimensional representations of the quantum algebra $U_{\mathbf{q}}(\widehat{sl(2)})$. The similar $U_{\mathbf{q}}(\widehat{sl(2)})$-structure can be observed in the quantum AKNS hierarchy. In consequence of this the quantum KdV and AKNS hierarchies share some common formal algebraic properties. In particular, the quantum transfer matrices $\mathbb{T}_j$ associated with the $2j + 1$ dimensional representations of $U_{\mathbf{q}}(\widehat{sl(2)})$ in both hierarchies are recursively expressed through the operators $\mathbb{T} \equiv \mathbb{T}_{\frac{1}{2}}$ and the unit operator $\mathbb{I} \equiv \mathbb{T}_0$ by means of the same fusion relation (see, e.g., [56]):

$$\mathbb{T}(\lambda)\mathbb{T}_j\left(\mathbf{q}^{j+\frac{1}{2}}\lambda\right) = \mathbb{T}_{j-\frac{1}{2}}(\mathbf{q}^{j+1}\lambda) + \mathbb{T}_{j+\frac{1}{2}}(\mathbf{q}^{j}\lambda). \tag{194}$$

Here $\mathbf{q}$ is the parameter of deformation of $U_{\mathbf{q}}(\widehat{sl(2)})$. For the AKNS hierarchy it should be chosen as

$$\mathbf{q} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}}. \tag{195}$$

In section 8.2 the vacuum eigenvalue of $\mathbb{T}$ was identified with the Wronskian (183). With the fusion relation (194) we can express now the all vacuum eigenvalues,

$$\mathbb{T}_j(\lambda)|\mathbf{P}\rangle = T_j^{(\mathrm{vac})}(\lambda)|\mathbf{P}\rangle, \tag{196}$$

in terms of solutions of the differential equation (145). The result appears to be in a remarkable form generalizing (183):

$$T_j^{(\mathrm{vac})}(\lambda) = \mathrm{i}(-1)^{2j+1}\mathrm{W}\left[\Psi_-\left(\theta + \frac{\pi\mathrm{i}(2j+1)}{n+2}, q; y\right), \Psi_-\left(\theta - \frac{\pi\mathrm{i}(2j+1)}{n+2}, q; y\right)\right]. \tag{197}$$

Recall that $\lambda = \left(\frac{\kappa_0}{n}\,\mathrm{e}^{\theta}\right)^{\frac{n+2}{n}}$, where the constant κ_0 is given by (169).

We cannot resist the temptation to mention here an evidence of existence of additional commuting family in the quantum AKNS hierarchy which does not have a classical counterpart.

Indeed, let us look at the solution Ψ_+ (140). In the complete analogy with the discussion from the previous subsection we consider Ψ_+ as a function of the variable $y = x + \log(2\kappa)$ and the parameters θ (174) and q, i.e., $\Psi_+ = \Psi_+(\theta, q; y)$. It is easy to see that the transformation

$$y \to y \pm \mathrm{i}\pi, \qquad \theta \to \theta \pm \frac{2\pi \mathrm{i} n}{n+2}, \qquad q \to -q \tag{198}$$

leaves equation (145) unchanged. Because of this the functions $\Psi_+\big(\theta \pm \frac{\mathrm{i}\pi n}{n+2}, -q; y \pm \mathrm{i}\pi\big)$ solve (145) and the following Wronskian does not depend on y:

$$\tilde{T}(\theta, q) = \mathrm{i}\mathrm{W}\left[\Psi_+\left(\theta - \frac{\mathrm{i}\pi n}{n+2}, -q; y - \mathrm{i}\pi\right), \Psi_+\left(\theta + \frac{\mathrm{i}\pi n}{n+2}, -q; y + \mathrm{i}\pi\right)\right]. \tag{199}$$

It is straightforward to show that $\tilde{T}(\theta, q)$ satisfies the relations:

$$W_q(\theta)\tilde{T}(\theta, q) = \mathrm{e}^{-\mathrm{i}\pi q} W_{-q}\left(\theta + \frac{\mathrm{i}\pi n}{n+2}\right) + \mathrm{e}^{\mathrm{i}\pi q} W_{-q}\left(\theta - \frac{\mathrm{i}\pi n}{n+2}\right), \tag{200}$$

and

$$\tilde{T}\left(\theta - \frac{2\mathrm{i}\pi}{n+2}, q\right) = \tilde{T}(\theta, q). \tag{201}$$

The evident similarity between (200), (201) and (183), (184) suggests interpreting $\tilde{T}(\theta, q)$ as a vacuum eigenvalue of 'dual' transfer matrix $\tilde{\mathbb{T}}$. Due to equation (201), it is expected that $\tilde{\mathbb{T}}$ admits a convergent power series expansion in terms of the 'dual' spectral parameter $\tilde{\lambda} = \kappa^{n+2}$. Using properties of the Wronskian (199), it is easy to guess a set of general conditions for the operator $\tilde{\mathbb{T}}(\tilde{\lambda})$ which is similar to that expounded in section 8.2 for $\mathbb{T}(\lambda)$. In all likelihood the appearance of 'dual' transfer matrix is related to the hidden $U_{\tilde{\mathbf{q}}}(\widehat{sl(2|1)})$-structure in the quantum AKNS hirarchy[17].

9. Infrared fixed point of the IPH boundary flow

In this work we have mainly focused our attention on the vacuum boundary amplitude (19). Of course, even the exact expression (142) does not define the boundary state completely. However, as we saw in section 8, some properties of the vacuum eigenvalue inherit important general features of the whole boundary state operator. Among them is the remarkable $T - Q$ equation which is a keystone relation in the theory of integrable systems. Another property of Z which should be understood in more general terms is the large κ expansion (167). It suggests that the boundary state operator $\mathbb{B}$ not only commutes with the local IM, but also admits an asymptotic expansion in terms of these operators,

$$\mathbb{B} \simeq \mathbb{B}_{\mathrm{IR}} \exp\left\{-\sum_{s=1}^{\infty} \frac{\mathbb{I}_s^{(\mathrm{norm})}}{2\sin\left(\frac{\pi s}{n+2}\right)} \left(\frac{n}{2E_*}\right)^s\right\}, \tag{202}$$

where the local IM $\mathbb{I}_s^{(\mathrm{norm})}$ are normalized in accordance with condition (191).

From the physical point of view the most interesting object to be discussed is the infrared boundary state associated with the operator $\mathbb{B}_{\mathrm{IR}}$ (202). According to our consideration this boundary state should have the form

$$|B\rangle_{\mathrm{IR}} = \int_{\mathbf{P}} \mathrm{d}^2\mathbf{P}\, Z_{\mathrm{IR}}^* |\mathcal{I}_{\mathbf{P}}\rangle, \tag{203}$$

[17] The screening operators (81), (99) can be interpreted as generators of the Borel subalgebra of the quantum affine superalgebra $U_{\tilde{\mathbf{q}}}(\widehat{sl(2|1)})$ with $\tilde{\mathbf{q}} = \mathrm{e}^{-\mathrm{i}\pi n}$.

where the amplitude Z_{IR} is given by equation (168) and therefore the states $|\mathcal{I}_{\mathbf{P}}\rangle \in \mathcal{F}_{\mathbf{P}} \otimes \bar{\mathcal{F}}_{\mathbf{P}}$ are normalized as follows:

$$\langle \mathbf{P}'|\mathcal{I}_{\mathbf{P}}\rangle = \delta(\mathbf{P}' - \mathbf{P}). \tag{204}$$

Of course, the boundary state (203) should obey the integrability condition (6). It is also expected to be a *conformal* boundary state which is some deformation to the domain $n > 0$ of the infrared boundary state with $n = 0$ described in section 6.2. In appendix C it is shown that a boundary state satisfying the above-mentioned conditions must also possess an extended conformal symmetry generated by the holomorphic spin-1 and spin-2 currents:

$$\big[W_s^{(\mathrm{IR})}(\tau) - \bar{W}_s^{(\mathrm{IR})}(\tau)\big]_{\sigma=0}|B\rangle_{\mathrm{IR}} = 0 \qquad (s = 1, 2), \tag{205}$$

where

$$W_1^{(\mathrm{IR})} = \partial X^\star, \qquad W_2^{(\mathrm{IR})} = -(\partial Y^\star)^2 + \frac{\mathrm{i}}{\sqrt{2}}\partial^2 Y^\star. \tag{206}$$

In equation (206) we use the notations

$$X^\star = \sqrt{\frac{n+2}{2}}X - \mathrm{i}\sqrt{\frac{n}{2}}Y, \qquad Y^\star = \sqrt{\frac{n+2}{2}}Y + \mathrm{i}\sqrt{\frac{n}{2}}X. \tag{207}$$

Note that the spin-2 current $W_2^{(\mathrm{IR})}$ produces the Virasoro algebra with centre charge $c = -2$. If

$$q^\star = \frac{\sqrt{n+2}}{2}Q + \mathrm{i}\frac{\sqrt{n}}{2}P \neq \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \tag{208}$$

the building blocks (Ishibashi states) $|\mathcal{I}_{\mathbf{P}}\rangle$ in (203) are defined by equations (204), (205) uniquely for the Fock spaces $\mathcal{F}_{\mathbf{P}} \otimes \bar{\mathcal{F}}_{\mathbf{P}}$ with the zero-mode momentum $\mathbf{P} = (P, Q)$ [3, 4]. In particular,

$$\begin{aligned}|\mathcal{I}_{\mathbf{P}}\rangle = \exp\left(\sum_{k=1}^{\infty}\frac{2}{k}X^\star_{-k}\bar{X}^\star_{-k}\right)\Bigg[1 &+ \frac{q^\star + \frac{1}{2}}{q^\star - \frac{1}{2}}2Y^\star_{-1}\bar{Y}^\star_{-1}\\ &+ \frac{q^\star + \frac{3}{2}}{q^\star - \frac{1}{2}}\left((Y^\star_{-1})^2 + \frac{1}{\sqrt{2}}Y^\star_{-2}\right)\left((\bar{Y}^\star_{-1})^2 + \frac{1}{\sqrt{2}}\bar{Y}^\star_{-2}\right)\\ &+ \frac{q^\star + \frac{1}{2}}{q^\star - \frac{3}{2}}\left((Y^\star_{-1})^2 - \frac{1}{\sqrt{2}}Y^\star_{-2}\right)\left((\bar{Y}^\star_{-1})^2 - \frac{1}{\sqrt{2}}\bar{Y}^\star_{-2}\right) + \cdots\Bigg]|\mathbf{P}\rangle.\end{aligned} \tag{209}$$

Here $X^\star_{-k}, Y^\star_{-k}$ are the oscillatory modes of the fields (207).

It can be shown that the integrand in equation (203), where $|\mathcal{I}_{\mathbf{P}}\rangle$ is given by (209), is well defined even for the exceptional values $q^\star = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ and therefore, we propose (203), (209) for the infrared boundary state of the IPH model. Note that it splits into the Dirichlet boundary state corresponding to the Dirichlet boundary condition $X^\star_B = \mathrm{const}$, and the n-independent conformal boundary state for the field $Y^\star$. Unfortunately, a consistent Lagrangian description of the $Y^\star$-component of $|B\rangle_{\mathrm{IR}}$ is still a question for the authors.

We would like to conclude the paper by a remark about the higher level boundary amplitudes $B_\alpha(\mathbf{P})$ in equation (13). It seems likely that they also admit a description in terms of ODEs similar to (138). Such higher level differential equations can be constructed along the line of [57] exploring analytical properties of the boundary state operator $\mathbb{B}$ as a function of variable θ (174). We intend to return to this problem in a future work.

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Appendix A

Here we present an explicit form of the first local IM of the AKNS series as an operator acting in the Fock space. Let us introduce the notations,

$$\mathbb{X}^{\mu_1\ldots\mu_m}_{i_1\ldots i_m} = \sum_{\substack{s_1+\ldots s_m=0\\ s_j\neq 0}} (s_1)^{i_1}\ldots(s_m)^{i_m} : X^{\mu_1}_{s_1}\ldots X^{\mu_m}_{s_m} : . \tag{A.1}$$

The normal ordering : : in this formula means that the operators X^i_s (89) with the bigger s are placed to the right. The local IM can be written as follows:

$$\mathbb{I}_1 = R^{-1}\big[\mathbb{X}^{11}_{00} + \mathbb{X}^{22}_{00} + I^{(\mathrm{vac})}_1(P,Q)\big], \tag{A.2}$$

$$\mathbb{I}_2 = R^{-2}\left[\frac{6n+4}{3}\mathbb{X}^{222}_{000} + 2n\mathbb{X}^{112}_{000} + (3n+2)Q\mathbb{X}^{22}_{00} + nQ\mathbb{X}^{11}_{00}\right.$$
$$\left. + 2nP\mathbb{X}^{12}_{00} - 2\mathrm{i}(n+1)\sqrt{n}\mathbb{X}^{12}_{10} + I^{(\mathrm{vac})}_2(P,Q)\right],$$

and

$$\begin{aligned}\mathbb{I}_3 = R^{-3}\Big[& n\mathbb{X}^{1111}_{0000} + (5n+4)\mathbb{X}^{2222}_{0000} + 6n\mathbb{X}^{1122}_{0000} + 2Pn\mathbb{X}^{111}_{000} \\ & + 2(5n+4)Q\mathbb{X}^{222}_{000} + 6nP\mathbb{X}^{122}_{000} + 6nQ\mathbb{X}^{112}_{000} - 6\mathrm{i}\sqrt{n}(n+1)\mathbb{X}^{122}_{100} \\ & + \frac{n}{2}(3P^2+3Q^2-1)\mathbb{X}^{11}_{00} + \frac{1}{2}(3(5n+4)Q^2 + 3nP^2 - 3n - 2)\mathbb{X}^{22}_{00} \\ & + 6nPQ\mathbb{X}^{12}_{00} - 6\mathrm{i}\sqrt{n}(n+1)Q\mathbb{X}^{12}_{10} \\ & - (n^2+3n+1)\mathbb{X}^{11}_{11} - (n^2+4n+2)\mathbb{X}^{22}_{11} + I^{(\mathrm{vac})}_3(P,Q)\Big].\end{aligned} \tag{A.3}$$

Here $I^{(\mathrm{vac})}_s(P,Q)$ are the vacuum eigenvalues of the AKNS integrals (94).

Appendix B

Here we prove the quantum Wronskian condition (175) and equation (184).

We start with an observation that the following transformations of the variables (y,κ,q),

$$\begin{aligned}\hat\Lambda:&\quad y\to y+\mathrm{i}\pi, &\quad \kappa \to \mathrm{e}^{\frac{\mathrm{i}\pi n}{n+2}}\kappa, &\quad q\to -q,\\ \hat\Omega:&\quad y\to y, &\quad \kappa\to \mathrm{e}^{\frac{2\mathrm{i}\pi}{n+2}}, &\quad q\to q,\end{aligned} \tag{B.1}$$

leave the ODE (145) unchanged while acting nontrivially on its solutions. The transformation $\hat\Lambda$ applied to the solution Ψ_+ (140) yields another solution, and the pair of functions

$$\Psi_+ = \Psi_+(y,\theta,q), \qquad \hat\Lambda\Psi_+ = \Psi_+\left(y+\mathrm{i}\pi, \theta + \frac{\mathrm{i}\pi n}{n+2}, -q\right), \tag{B.2}$$

with θ given by (174), forms a basis in the space of solutions of (145). It is not difficult to check that

$$W[\Psi_+, \hat{\Lambda}\Psi_+] = e^{i\pi(q-\frac{1}{2})}, \tag{B.3}$$

i.e., the solutions (B.2) are indeed linearly independent. The solution Ψ_- (139) can always be expanded in this basis, in particular

$$\Psi_- = a\Psi_+ + b\hat{\Lambda}\Psi_+. \tag{B.4}$$

Using equation (B.3) we conclude that

$$b = e^{i\pi(\frac{1}{2}-q)} W_q(\theta), \tag{B.5}$$

where $W_q(\theta) = W\big[\Psi_+, \Psi_-\big]$. The transformation $\hat{\Lambda}$ leaves the solution Ψ_- unchanged[18], i.e., $\Psi_-(y, \theta, q) = \Psi_-\big(y + i\pi, \theta + \frac{i\pi n}{n+2}, -q\big)$. This allows one to express the coefficient a in (B.4) in terms of the Wronskian $W_q(\theta)$ as well:

$$a = -e^{i\pi(\frac{1}{2}-q)} W_{-q}\left(\theta + \frac{i\pi n}{n+2}\right). \tag{B.6}$$

Let us apply now the transformation $\hat{\Omega}$ (B.1) to the both sides of equation (B.4). It is apparent that Ψ_+ is invariant with respect to the action of $\hat{\Omega}$, and hence,

$$\hat{\Omega}\Psi_- = e^{i\pi(\frac{1}{2}-q)}\left(W_q\left(\theta + \frac{2i\pi}{n+2}\right)\hat{\Lambda}\Psi_+ - W_{-q}(\theta + i\pi)\Psi_+\right). \tag{B.7}$$

The quantum Wronskian condition (175) follows immediately from (B.4), (B.7), (B.3) and the relation

$$W[\Psi_-, \hat{\Omega}\Psi_-] = i. \tag{B.8}$$

It is easy to derive now from the last equation that three solutions Ψ_-, $\hat{\Omega}\Psi_-$ and $\hat{\Omega}^{-1}\Psi_-$ satisfy the relation,

$$T(\theta, q)\Psi_- = \hat{\Omega}\Psi_- + \hat{\Omega}^{-1}\Psi_-, \tag{B.9}$$

with function $T(\theta, q)$ defined by equation (183). Taking the Wronskian from the both sides of this equation with the solution $\Psi_+ = \hat{\Omega}\Psi_+ = \hat{\Omega}^{-1}\Psi_+$ we arrive at equation (184).

Appendix C

Here we show that the infrared boundary state of the IPH model, $|B\rangle_{\rm IR}$, satisfies conditions (205)–(207).

Our analyses are based on the assumption that $|B\rangle_{\rm IR}$ possesses the conformal symmetry, i.e.:

$$[T^{(\rm IR)}(\tau) - \bar{T}^{(\rm IR)}(\tau)]_{\sigma=0}|B\rangle_{\rm IR} = 0, \tag{C.1}$$

where the holomorphic and antiholomorphic components of infrared stress–energy tensor have the most general admissible form

$$T^{(\rm IR)} = -\partial\mathbf{X}\cdot\partial\mathbf{X} + i\boldsymbol{\rho}\cdot\partial^2\mathbf{X}, \qquad \bar{T}^{(\rm IR)} = -\bar{\partial}\mathbf{X}\cdot\bar{\partial}\mathbf{X} + i\boldsymbol{\rho}\cdot\bar{\partial}^2\mathbf{X}. \tag{C.2}$$

The constant vector $\boldsymbol{\rho} = (\rho_x, \rho_y)$ is unknown *a priori*, but it is expected to be a function of coupling constant n. A comparison between (C.2) and the infrared stress–energy tensor for $n = 0$ (137) shows that

$$\rho_y|_{n=0} = \frac{1}{\sqrt{2}}. \tag{C.3}$$

[18] This property of Ψ_- leads immediately to equation (185).

The local fields (C.2) and the first nontrivial local IM $\mathbb{I}_2$ ($\bar{\mathbb{I}}_2$) (A.2) allow one to construct holomorphic spin-3 and antiholomorphic spin-(-3) currents through the relations

$$\partial W_3^{(\mathrm{IR})} = [T, \mathbb{I}_2], \qquad \bar{\partial} \bar{W}_3^{(\mathrm{IR})} = [\bar{T}, \bar{\mathbb{I}}_2]. \tag{C.4}$$

Since the infrared boundary state satisfies both equation (C.1) and the integrability condition (7), it is apparent that

$$\left[W_3^{(\mathrm{IR})}(\tau) - \bar{W}_3^{(\mathrm{IR})}(\tau)\right]_{\sigma=0} |B\rangle_{\mathrm{IR}} = 0. \tag{C.5}$$

An explicit form of $W_3^{(\mathrm{IR})}$ is not particularly important for the present discussion. It is important that this field and $T^{(\mathrm{IR})}$ generate a spin-1 current $W_1^{(\mathrm{IR})}$ through their operator product expansion:

$$T^{(\mathrm{IR})}(u) W_3^{(\mathrm{IR})}(v) = \frac{6 W_1^{(\mathrm{IR})}(v)}{(u-v)^4} + \frac{2\partial W_1^{(\mathrm{IR})}(v)}{(u-v)^3} + O((u-v)^{-2}). \tag{C.6}$$

Explicit calculations show that $W_1^{(\mathrm{IR})}$ has the form

$$W_1^{(\mathrm{IR})} = \alpha \partial X + \mathrm{i}\beta \partial Y, \tag{C.7}$$

with

$$\alpha = 2\rho_y \sqrt{n}(1 + n + \mathrm{i}\rho_x \sqrt{n}), \qquad \beta = n\rho_x^2 + \rho_y^2(2+3n) + 2\mathrm{i}\rho_x(1+n)\sqrt{n} - 2n - 1. \tag{C.8}$$

As it follows from equations (C.8) the current $W_1^{(\mathrm{IR})}$ may vanish for $\rho_y = 0$ or $\rho_y = \pm\sqrt{n+2}$ only. Since none of these is consistent with equation (C.3), the infrared boundary state should satisfy condition (205) for the nonvanishing spin-1 current given by equations (C.7) and (C.8).

Let us consider now the operator product expansion $W_1^{(\mathrm{IR})}(u) W_3^{(\mathrm{IR})}(v)$. It contains a singular term $\sim (u-v)^{-2}$ which involves the spin-2 current of the form,

$$\tilde{W}_2^{(\mathrm{IR})} = 2\mathrm{i}n\beta(\partial X)^2 + 4n\alpha \partial X \partial Y + \mathrm{i}\beta(6n+4)(\partial Y)^2 + \gamma \partial^2 X + \delta \partial^2 Y, \tag{C.9}$$

where

$$\begin{aligned} \gamma &= n(\beta\rho_x - \mathrm{i}\alpha\rho_y) + \mathrm{i}(n+1)\beta\sqrt{n}, \\ \delta &= -\alpha\sqrt{n}(n + 1 + \mathrm{i}\rho_x\sqrt{n}) + \beta\rho_y(3n+2). \end{aligned} \tag{C.10}$$

With equations (C.7), (C.9) it is easy to see that the operator product expansion $W_1^{(\mathrm{IR})}(u) \tilde{W}_2^{(\mathrm{IR})}(v)$ produces one more spin-1 current

$$\tilde{W}_1^{(\mathrm{IR})} = 2n\mathrm{i}\alpha\beta \partial X + (n\alpha^2 - (3n+2)\beta^2)\partial Y. \tag{C.11}$$

This current does not vanish provided $W_1^{(\mathrm{IR})}$ exists. Therefore one needs to explore two possibilities: either $W_1^{(\mathrm{IR})}$ and $\tilde{W}_1^{(\mathrm{IR})}$ are linear independent or linear dependent currents. The first possibility implies that $|B\rangle_{\mathrm{IR}}$ is the n-independent Dirichlet boundary state associated with the Dirichlet boundary condition $\mathbf{X}_B = \mathrm{const}$. It should be apparently ignored. Hence $\tilde{W}_1^{(\mathrm{IR})} \sim W_1^{(\mathrm{IR})}$. With equations (C.7), (C.11) it gives the relation

$$\sqrt{n}\alpha \pm \sqrt{n+2}\beta = 0. \tag{C.12}$$

Now it is convenient to introduce a new set $(X^\star, Y^\star)$ related to the basic fields (X, Y) through the (complex) orthogonal transformation:

$$X^\star = \sqrt{\frac{n+2}{2}} X \mp \mathrm{i}\sqrt{\frac{n}{2}} Y, \qquad Y^\star = \mathrm{i}\sqrt{\frac{n}{2}} X \pm \sqrt{\frac{n+2}{2}} Y. \tag{C.13}$$

Here the sign factors are dictated by the choice of sign in (C.12). Since $W_1^{(\mathrm{IR})} \sim \partial X^\star$ one can choose (without loss of generality) the infrared stress–energy tensor (C.2) in the form

$$T^{(\mathrm{IR})} = -(\partial X^\star)^2 + W_2^{(\mathrm{IR})}, \tag{C.14}$$

where

$$W_2^{(\mathrm{IR})} = -(\partial Y^\star)^2 + \frac{\mathrm{i}\sigma}{\sqrt{2}}\partial^2 Y^\star. \tag{C.15}$$

It follows immediately from equations (C.8) and (C.12) that $\sigma^2 = 1$.

Thus there are four nonvanishing spin-2 currents, $T^{(\mathrm{IR})}$, $W_2^{(\mathrm{IR})}$, $\tilde{W}_2^{(\mathrm{IR})}$ and $\partial^2 X^\star$, satisfying the condition $(W_2 - \bar{W}_2)|B\rangle_{\mathrm{IR}} = 0$. If one admits now their linear independence, then $\partial^2 Y^\star$ can be expressed in terms of these currents and, consequently, should also satisfy the above condition. This will eventually lead us to the Dirichlet boundary state corresponding to $\mathbf{X}_B = \mathrm{const}$. Therefore we reject this possibility and accept that the current $\tilde{W}_2^{(\mathrm{IR})}$ is linearly expressed in terms of $T^{(\mathrm{IR})}$, $W_2^{(\mathrm{IR})}$ and $\partial^2 X^\star$. It is straightforward to check that this is indeed the case provided

$$\sigma = +1. \tag{C.16}$$

The remaining ambiguity in sign factors in equations (C.13) should be resolved in accordance with the condition (C.3). In this way one arrives at equations (205)–(207).

Finally we note that the local IM $\mathbb{I}_2$ (A.2) can be expressed in terms of the fields (206), (207) as follows

$$\mathbb{I}_2 = (2n+1)\sqrt{\frac{n+2}{2}}\,\mathbb{I}_2^{(n=0)}[Y^\star] - 2\sqrt{2n}\,\mathbb{J}_2\big[W_1^{(\mathrm{IR})}, W_2^{(\mathrm{IR})}\big], \tag{C.17}$$

where

$$\mathbb{I}_2^{(n=0)}[Y^\star] = \frac{4\mathrm{i}}{3}\int_0^{2\pi R}\frac{\mathrm{d}\tau}{2\pi}(\partial Y^\star)^3, \tag{C.18}$$

and

$$\mathbb{J}_2[W_1, W_2] = \int_0^{2\pi R}\frac{\mathrm{d}\tau}{2\pi}\Big((n+1)W_2W_1 + \frac{2n+3}{3}(W_1)^3\Big). \tag{C.19}$$

Evidently $(\mathbb{J}_2 - \bar{\mathbb{J}}_2)|B\rangle_{\mathrm{IR}} = 0$; therefore, the integrability condition $(\mathbb{I}_2 - \bar{\mathbb{I}}_{-2})|B\rangle_{\mathrm{IR}} = 0$ for the boundary state (203), (209) with $n > 0$ follows from the fact that it holds for $n = 0$.

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